QUESTION

(a) If f is a function of u and v, where $u = x^2 - y^2$ and $v = xy$, use the chain rule to show that

$$\frac{\partial f}{\partial x} = 2x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v},$$

and find the corresponding expression for $\dfrac{\partial^2 f}{\partial y \partial x}$.

(b) Use Maclaurin's theorem to show that

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + R_3,$$

and write down the Lagrange form of R_3.

ANSWER

(a) $f = f(u, v)$

$$u = x^2 - y^2, \ \frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = -2y, \qquad v = xy, \ \frac{\partial v}{\partial x} = y, \ \frac{\partial v}{\partial y} = x.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
&= 2x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right) + \frac{\partial f}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right) \\
&= 2x\left\{\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial y}\right\} + \frac{\partial f}{\partial v} + y\left\{\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right\} \\
&= 2x\left\{\frac{\partial^2 f}{\partial u^2}(-2y) + x\frac{\partial^2 f}{\partial v \partial u}\right\} + \frac{\partial f}{\partial v} + y\left\{-2y\frac{\partial^2 f}{\partial u \partial v} + x\frac{\partial^2 f}{\partial v^2}\right\} \\
\frac{\partial^2 f}{\partial y \partial x} &= -4xy\frac{\partial^2 f}{\partial u^2} + 2(x^2 - y^2)\frac{\partial^2 f}{\partial u \partial v} + xy\frac{\partial^2 f}{\partial v^2}
\end{aligned}$$

(b)

$$\begin{aligned}
&f(x) = (1+x)^{\frac{1}{3}} && f(0) = 1^{\frac{1}{3}} = 1 \\
&f'(x) = \tfrac{1}{3}(1+x)^{-\frac{2}{3}} && f'(0) = \tfrac{1}{3}(1)^{-\frac{2}{3}} = \tfrac{1}{3} \\
&f''(x) = \tfrac{1}{3}(-\tfrac{2}{3})(1+x)^{-\frac{5}{3}} && f''(0) = -\tfrac{2}{9}(1)^{-\frac{5}{3}} = -\tfrac{2}{9} \\
&f'''(x) = -\tfrac{2}{9}(-\tfrac{5}{3})(1+x)^{-\frac{8}{3}} && f'''(0) = \tfrac{10}{27}(1)^{-\frac{8}{3}} = \tfrac{10}{27} \\
&f^{(4)}(x) = \tfrac{10}{27}(-\tfrac{8}{3})(1+x)^{-\frac{11}{3}} &&
\end{aligned}$$

Substituting these into Maclaurins theorem gives

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \left(-\frac{2}{9}\right)\frac{x^2}{2!} + \frac{10}{27}\frac{x^3}{3!} + R_3,$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + R_3,$$

where $R_3 = \dfrac{x^4}{4!}f^{(4)}(c) = \dfrac{x^4}{24}\left(-\dfrac{80}{81}\right)(1+c)^{-\frac{11}{3}} = -\dfrac{10x^4}{243(1+c)^{\frac{11}{3}}}$

$0 < c < x$.