QUESTION

Find the eigenvalues of the matrix

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 3 & 3 \\ 0 & 2 & 4 \end{pmatrix},$$

and determine the corresponding eigenvectors.

ANSWER

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 3 & 3 \\ 0 & 2 & 4 \end{pmatrix}$$

Eigenvalues satisfy $|A - \lambda I| = 0$

$$\begin{aligned}
\begin{vmatrix} 2-\lambda & -1 & 2 \\ 0 & 3-\lambda & 3 \\ 0 & 2 & 4-\lambda \end{vmatrix} &= (2-\lambda)\{(3-\lambda)(4-\lambda)-6\} \\
&= (2-\lambda)\{12 - 7\lambda + \lambda^2 - 6\} \\
&= (2-\lambda)(6 - 7\lambda + \lambda^2) \\
&= (2-\lambda)(\lambda - 6)(\lambda - 1)
\end{aligned}$$

This is zero if $\lambda = 1, 2, 6$ so these are the eigenvalues.

$\lambda = 1:$ Eigenvector satisfies $(A - 1I)\mathbf{x} = 0$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
x_1 - x_2 + 2x_3 &= 0 \\
2x_2 + 3x_3 &= 0 \\
2x_2 + 3x_3 &= 0
\end{aligned}$$

Choose $x_3 = c, \Rightarrow x_2 = -\dfrac{3c}{2}, \Rightarrow x_1 = x_2 - 2x_3 = -\dfrac{3c}{2} - 2c = -\dfrac{7c}{2}$

Therefore the eigenvector is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{7c}{2} \\ -\frac{3c}{2} \\ c \end{pmatrix}$, where c is any constant.

$\lambda = 6:$ Eigenvector satisfies $(A - 6I)\mathbf{x} = 0$

$$\begin{pmatrix} -4 & -1 & 2 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -4x_1 - x_2 + 2x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \\ 2x_2 - 2x_3 &= 0 \end{aligned}$$

Choose $x_2 = d, \Rightarrow x_3 = d, \Rightarrow 4x_1 = 2x_3 - x_2 = 2d - d = d,\ x_1 = \dfrac{d}{4}$

Therefore the eigenvector is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{d}{4} \\ d \\ d \end{pmatrix}$, where d is any constant.

$\lambda = 2:$ Eigenvector satisfies $(A - 2I)\mathbf{x} = 0$

$$\begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -x_2 + 2x_3 &= 0 \\ x_2 + 3x_3 &= 0 \\ 2x_2 + 2x_3 &= 0 \end{aligned}$$

Hence $x_3 = 0,\ x_2 = 0$, choose $x_1 = e$.

Therefore the eigenvector is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix}$ where e is any constant.