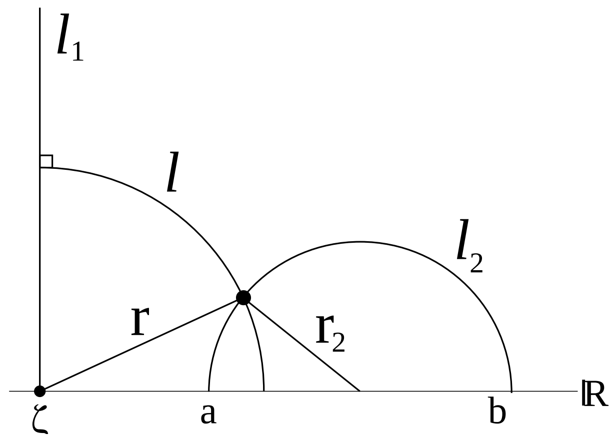
Question

Let ℓ_1 and ℓ_2 be parallel hyperbolic lines in $\mathbf{H}$, where ℓ_1 is contained in a vertical Euclidean line. Prove that ℓ_1 and ℓ_2 are ultraparallel if and only if there is a hyperbolic line ℓ perpendicular to both ℓ_1 and ℓ_2.

Answer



One way to proceed is by cases. Suppose that ℓ_2 is ultraparallel to ℓ_1 and has endpoints a, b ($a > 0, b > a$ as drawn. The case that $b < a < 0$ is similar). Any line perpendicular to ℓ_1 is contained in a euclidean circle centred at ξ (where ℓ_1 'intersects ' $\mathbf{R}$).

Such a line is perpendicular to ℓ_2 if and only if

$$r^2 + r_2^2 = \left(\frac{1}{2}(b+a) - \xi\right)^2$$

where r_2 is the radius of the circle containing ℓ_2 and r is the radius of the circle containing ℓ (and hence the only variable in the equation). That is

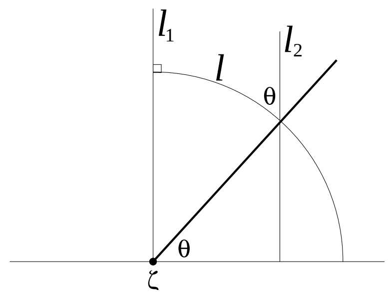
$$r = \sqrt{\left(\frac{1}{2}(b+a) - \xi\right)^2 - r_2^2}$$

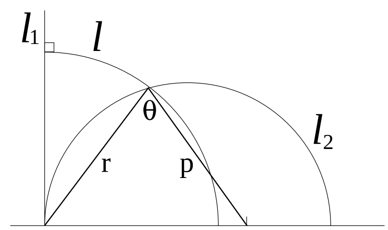
$\left(\text{and since } \ell_1 \cap \ell_2 = \emptyset (\ell_1, \ell_2 \text{ are disjoint}) \ \frac{1}{2}(\mathrm{b}+\mathrm{a}) - \xi > \mathrm{r}_2\right)$

So, such a circle ℓ exists, center ξ, radius r as above.

If $\ell_1 \ell_2$ are parallel but not ultraparallel, then either ℓ_2 is a vertical euclidean line or is a euclidean circle passing through ξ.

In the former case, no circle perpendicular to ℓ_1 can also be perpendicular to ℓ_2, since the angle between ℓ and ℓ_2 is equal to the argument of the point of intersection of ℓ and ℓ_2 (as shown in the picture).





We may in the latter case use the law of cosines to calculate the angle between ℓ (a circle perpendicular to ℓ_1 with radius r) and ℓ_2 (with fixed center c and fixed radius p to see that
$(c-\xi)^2 = r^2 + p^2 - 2rp\cos\theta$
$(c-\xi)^2 - p^2 = r^2 - 2p\cos\theta \cdot r$
The only way that $\theta = \dfrac{\pi}{2}$ is that

$$(c-\xi)^2 = r^2 + p^2$$

But note that $c - \xi = p$ (since $\ell_1\ell_2$ are parallel) and so $r = 0$ which is not a circle.⊗
So if $\ell_1\ell_2$ are ultraparallel there is a (unique) circle (containing a hyperbolic line) perpendicular to both. If $\ell_1\ell_2$ are parallel but not ultraparallel, no such circle exists and so we are done.