Question

Calculate the eigenvalues and the corresponding eigenvectors for each of the following matrices:

$$\text{(i)} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}; \quad \text{(ii)} \begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix}; \quad \text{(iii)} \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix};$$

$$\text{(iv)} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}; \quad \text{(v)} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Answer

(i) $\begin{vmatrix} 1-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 1, 2.$

$\underline{\lambda = 1}$ Solve $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $y = 0$

Let $x = \alpha,\ y = 0$ so a suitable eigenvector $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

$\underline{\lambda = 2}$ Solve $\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $-x - y = 0 \Rightarrow y = -x$

Let $x = \beta,\ y = -\beta$ so a suitable eigenvector $\begin{pmatrix} \beta \\ -\beta \end{pmatrix}$

(ii) $\begin{vmatrix} 3-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) + 5 = \lambda^2 - 5\lambda + 11 = 0$

$$\lambda = \frac{5 \pm \sqrt{25-44}}{2} = \frac{5 \pm \sqrt{19}i}{2}$$

No real roots, so no real eigenvectors.

(iii) $\begin{vmatrix} 4-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (4-\lambda)(-2-\lambda) + 1 = \lambda^2 - 2\lambda - 7 = 0$

$$\lambda = \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$$

$\underline{\lambda = 1 + 2\sqrt{2}}$

Solve $\begin{pmatrix} 3-2\sqrt{2} & 1 \\ -1 & -3-2\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $\left.\begin{array}{rcl} (3-2\sqrt{2})x+y & = & 0 \\ -x-(3+2\sqrt{2})y & = & 0 \end{array}\right\}$

Let $x=\alpha$, so $y=-\alpha(3-2\sqrt{2})$

Note from second equation:

$$y=\frac{-\alpha}{3+2\sqrt{2}}=\frac{-\alpha(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})}=\frac{-\alpha(3-2\sqrt{2})}{1}$$

Suitable eigenvector: $\begin{pmatrix} \alpha \\ \alpha(2\sqrt{2}-3) \end{pmatrix}$.

$\underline{\lambda=1-2\sqrt{2}}$

Solve $\begin{pmatrix} 3+2\sqrt{2} & 1 \\ -1 & -3+2\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $\left.\begin{array}{rcl} (3+2\sqrt{2})x+y & = & 0 \\ -x+(-3+2\sqrt{2})y & = & 0 \end{array}\right\}$

Let $x=\beta$, so $y=-\beta(3+2\sqrt{2})$

Suitable eigenvector: $\begin{pmatrix} \beta \\ -\beta(3+2\sqrt{2}) \end{pmatrix}$.

(iv) $\begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix}=(\cos\theta-\lambda)^2+\sin^2\theta=$

$\lambda^2-2\lambda\cos\theta+\cos^2\theta+\sin^2\theta=\lambda^2-2\lambda\cos\theta+1=0$

$$\lambda=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\cos\theta\pm i\sin\theta.$$

If $\sin\theta\neq 0$, then no real solutions, so no real eigenvectors.

If $\sin\theta=0$, then repeated eigenvalue $\lambda=\cos\theta$ with eigenvector equation

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. $\left.\begin{array}{lcl} \text{any x} & \Rightarrow & x=\alpha \\ \text{any y} & \Rightarrow & y=\beta \end{array}\right\}$ eigenvector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

(So any non-zero vector in the xy-plane is a suitable eigenvector)

(v) $\begin{vmatrix} \cos 2\theta - \lambda & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta - \lambda \end{vmatrix} = \lambda^2 - \cos^2 2\theta - \sin^2 2\theta = \lambda^2 - 1 = 0$

so $\lambda = \pm 1$.

<u>Assume $\sin 2\theta \neq 0$</u>

<u>$\lambda = 1$</u> Solve $\begin{pmatrix} \cos 2\theta - 1 & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} (\cos 2\theta - 1)x + (\sin 2\theta)y &= 0 \\ (\sin 2\theta)x - (\cos 2\theta + 1)y &= 0 \end{aligned}\right\}$$

Let $x = \alpha$ so $y = \dfrac{(1 - \cos 2\theta)\alpha}{\sin 2\theta}$ $(\sin 2\theta \neq 0)$

Suitable eigenvector: $\begin{pmatrix} \alpha \\ \frac{(1-\cos 2\theta)\alpha}{\sin 2\theta} \end{pmatrix}$

Note from second equation:

$$\begin{aligned} y = \frac{\alpha(\sin 2\theta)}{\cos 2\theta + 1} &= \frac{\alpha(\sin 2\theta)}{\cos 2\theta + 1}\left(\frac{1 - \cos 2\theta}{1 - \cos 2\theta}\right) \\ &= \frac{\alpha(\sin 2\theta)(1 - \cos 2\theta)}{1 - \cos^2 2\theta} \\ &= \frac{\alpha(\sin 2\theta)(1 - \cos 2\theta)}{\sin^2 2\theta} \\ &= \frac{(1 - 2\cos 2\theta)\alpha}{\sin 2\theta} \end{aligned}$$

<u>$\lambda = -1$</u> Solve $\begin{pmatrix} \cos 2\theta + 1 & \sin 2\theta \\ \sin 2\theta & \cos 2\theta + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} (\cos 2\theta + 1)x + (\sin 2\theta)y &= 0 \\ (\sin 2\theta)x + (1 - \cos 2\theta)y &= 0 \end{aligned}\right\}$$

Let $x = \beta$ so $y = \dfrac{-\beta(1 + \cos 2\theta)}{\sin 2\theta}$ $(\sin 2\theta \neq 0)$

Suitable eigenvector: $\begin{pmatrix} \beta \\ \frac{-\beta(1+\cos 2\theta)}{\sin 2\theta} \end{pmatrix}$

<u>If $\sin 2\theta = 0$</u>

The matrix becomes $\begin{pmatrix} \cos 2\theta & 0 \\ 0 & -\cos 2\theta \end{pmatrix}$ with eigenvalues

$$(\cos 2\theta - \lambda)(-\cos 2\theta - \lambda) = \lambda^2 - \cos^2 2\theta = 0 \Rightarrow \lambda = \pm\cos 2\theta.$$

$\underline{\lambda = \cos 2\theta}$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2\cos 2\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so } (-2\cos 2\theta)\text{y} = 0 \text{ and y} = 0$$

If $x = \alpha$ then suitable eigenvector: $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

$\underline{\lambda = -\cos 2\theta}$

$$\begin{pmatrix} 2\cos 2\theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so } (2\cos 2\theta)\text{x} = 0 \text{ and x} = 0$$

If $y = \beta$ then suitable eigenvector: $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$