Question
For small $|\varepsilon|$ show that the roots of the equation

$$x^3 - (2+\varepsilon)x^2 + (1-\varepsilon)x + 2 + 3\varepsilon = 0$$

are given by

$$x = \left\{ \begin{array}{l} 1 + \frac{3}{2}\varepsilon + O(\varepsilon^2) \\ -1 - \frac{1}{6}\varepsilon + O(\varepsilon^2) \\ 2 - \frac{1}{3}\varepsilon + o(\varepsilon^2) \end{array} \right\}$$

If ε is switched on from 0 such that $\varepsilon = \alpha i,\ 0 < \alpha <\lll 1$, sketch how the solutions move in the complex plane.

Answer
Use hint that $x = x_0 + x_1\varepsilon + O(\varepsilon^2)$ is a good ansatz.
Substitute:

(A)

$$\begin{aligned} x_3 &= [x_0 + x_1\varepsilon + O(\varepsilon^2)]^3 \\ &= x_0^3 + x_1^3\varepsilon^3 + O(\varepsilon^9) + 3x_0^2x_1\varepsilon + O(\varepsilon^2) + 3x_0x_1^2\varepsilon^2 + O(\varepsilon^4) \\ &= x_0^3 + 3x_0^2x_1\varepsilon + O(\varepsilon^2) \end{aligned}$$

We only want answers to $O(\varepsilon^2)$.

(B)

$$\begin{aligned} -(2+\varepsilon)x^2 &= -(2+\varepsilon)(x_0 + \varepsilon x_1 + O(\varepsilon^2)) \\ &= -(2+\varepsilon)(x_0^2 + 2x_0x_1\varepsilon + O(\varepsilon^2)) \\ &= -2x_0^2 - 4x_0x_1\varepsilon - \varepsilon x_0^2 + O(\varepsilon^2) \end{aligned}$$

(C)

$$\begin{aligned} -(1-\varepsilon)x &= -(1-\varepsilon)(x_0 + \varepsilon x_1 + O(\varepsilon^2)) \\ &= -(x_0 + \varepsilon x_1 - \varepsilon x_0 + O(\varepsilon^2)) \end{aligned}$$

(D)

$$2 + 3\varepsilon = 2 + 3\varepsilon$$

$$\text{(A)} + \text{(B)} + \text{(C)} + \text{(D)} = (\mathrm{x}_0^3 - 2\mathrm{x}_0^2 - \mathrm{x}_0 + 2)$$

$$+\varepsilon(3x_0^2x_1 - 4x_0x_1 + x_0^2 - x_1 + x_0 + 3) + O(\varepsilon^2) = 0 \text{ (original equation!)}$$

Therefore at $O(\varepsilon^0) : x_0^3 - 2x_0^2 - x_0 + 2 = 0$ (unperturbed equation)

Use hints in question to observe solutions are $x_0 = \pm 1,\ 2$.

Therefore

$O(\varepsilon^1) : 3x_0^2 x_1 - 4x_0 x_1 - x_0^2 - x_1 + x_0 + 3 = 0$

$$x_1 = \frac{(3 + x_0 - x_0^2)}{(1 + 4x_0 - 3x_0^2)}$$

$$\left.\begin{array}{lll} x_0 = 1 & \Rightarrow & x_1 = \frac{3}{2} \\ x_0 = -1 & \Rightarrow & x_1 = -\frac{1}{6} \\ x_0 = 2 & \Rightarrow & x_1 = -\frac{1}{3} \end{array}\right\} x = \left\{\begin{array}{l} +1 + \frac{3}{2}\varepsilon + O(\varepsilon^2) \\ -1 - \frac{1}{6}\varepsilon + O(\varepsilon^2) \\ 2 - \frac{1}{3}\varepsilon + O(\varepsilon^2) \end{array}\right.$$

(x)