**Question**

Sketch the graphs of $\cot x$ and $\frac{1}{x}$. Show that the solutions of $x \cot x = 1$ behave like

$$x = (1 + \frac{1}{2})\pi - \frac{1}{(n+\frac{1}{2})\pi} + \cdots, \quad n \text{ integer.}$$

**Answer**

PICTURE

Look for $x \cot x = 1 \Rightarrow \cot x = \frac{1}{x}$.

As $x \to \infty$ $\frac{1}{x} \to 0$

Therefore if it crosses $\cot x$ it does so near to $\cot x = 0$.

$\cot x = 0$ at $x = \left(n + \frac{1}{2}\right)\pi$ where $n$ is an integer.

(since $\tan\left(n + \frac{1}{2}\right)\pi \to \pm\infty$)

Therefore first guess is $x = \left(n + \frac{1}{2}\right)\pi + \delta$ where $\delta \ll \left(n + \frac{1}{2}\right)\pi$

$$\begin{aligned}\cot\left[\left(n+\frac{1}{2}\right)\pi+\delta\right] &= \frac{1}{\tan((n+\frac{1}{2}\pi)+\delta)}\\ &= \frac{1-\tan(n+\frac{1}{2})\pi\tan\delta}{\tan(n+\frac{1}{2})\pi+\tan\delta}\\ &= -\tan\delta\end{aligned}$$

Therefore $-\tan\delta = \dfrac{1}{(n+\frac{1}{2})\pi+\delta}$ is the equation to solve for $\delta$.

$$\left.\begin{aligned}&\tan\delta = \delta + \frac{\delta^3}{3} + O(\delta^5)\\ &\text{Therefore } -\delta - \frac{\delta^3}{3} + \mathrm{O}(\delta^5) = \frac{1}{(\mathrm{n}+\frac{1}{2})\pi+\delta}\\ &\Rightarrow -\left(n+\frac{1}{2}\right)\pi\delta + O(\delta^2) = 1\\ &\Rightarrow \delta \approx -\frac{1}{(n+\frac{1}{2})\pi}\end{aligned}\right\}$$

so $x = \left(n+\dfrac{1}{2}\right)\pi - \dfrac{1}{(n+\frac{1}{2})\pi} + \cdots \quad n\to\infty$ (hence $x\to\infty$)