Question

Show that the small ε expansion of the roots of

$$x^3-(3+\varepsilon)x-2+\varepsilon=0$$

are given by

$$x=\begin{array}{l}2+\dfrac{1}{9}\varepsilon+O(\varepsilon^2)\\ -1\pm\sqrt{\dfrac{2}{3}\varepsilon}+O(\varepsilon)\end{array}$$

Sketch the behaviour of the roots as $\varepsilon\to 0^+$.

Answer

$x^3-(3_+\varepsilon)x-2+\varepsilon=0$
Put $\varepsilon=0$
$x^3-3x-2=0\to$ obvious root $x=-1$
Therefore

$$\left.\begin{array}{rcl}(x+1)(x^2-x-2)&=&0\\(x+1)(x+1)(x-2)&=&0\end{array}\right\}\text{ so } x=-1 \text{ \underline{twice} and } 2$$

Therefore 2 degenerate roots at $\varepsilon=0$ so try
$x=x_0+\varepsilon^{\frac{1}{2}}x_1+O(\varepsilon)$ to capture these.
Other root is regular so can use $x=x_0+\varepsilon x_1+O(\varepsilon^2)$.
Substitute:
$[x_0+\varepsilon^{\frac{1}{2}}x_1+O(\varepsilon)]^3-(3+\varepsilon)(x_0+\varepsilon^{\frac{1}{2}}x_1+O(\varepsilon))-2+\varepsilon=0$
$x_0^3+3x_0^2x_1\varepsilon^{\frac{1}{2}}-3x_0-3x_1\varepsilon^{\frac{1}{2}}-2+O(\varepsilon)=0$
Balance at
$O(\varepsilon^0):x_0^3-3x_0-2=0\Rightarrow x_0=-1$ twice as above (and 2 but this gives regular root)
$O(\varepsilon^{\frac{1}{2}}):3x_0^2x_1-3x_1=0\Rightarrow 0\cdots x_1=0,\ \underline{x_1=?}$
Can't find x_1 at this order. Need to go to $O(\varepsilon)$.
After more algebra:
$O(\varepsilon):1-x_0+3x_0^2x_2-3x_2+3x_0x_1^2=0$
$\Rightarrow 2-3x_1^2=0\Rightarrow x_1=\pm\sqrt{\dfrac{2}{3}}$. ($x_2$ got from $O(\varepsilon^{\frac{3}{2}})$ etc.)

Therefore degenerate root splits as: $x=-1\pm\sqrt{\dfrac{2\varepsilon}{3}}+O(\varepsilon)$

Other root: $x=x_0+\varepsilon x_1+O(\varepsilon^2)$
$[x_0+\varepsilon x_1+O(\varepsilon^2)]^3-(3+\varepsilon)(x_0+\varepsilon x_1+O(\varepsilon^2))-2+\varepsilon=0$
$x_0^3+3x_0^2x_1\varepsilon-3x_0-3x_1\varepsilon-\varepsilon x_0-2+\varepsilon+O(\varepsilon^2)=0$

Balance at:

$O(\varepsilon^0) : x_0^3 - 3x_0 - 2 = 0 \rightarrow$ roots as above, pick $x_0 = 2$

$O(\varepsilon^1) : 3x_0^2 x_1 - 3x_1 - x_0 + 1 = 0 \rightarrow x_0 = 2 \Rightarrow 9x_1 - 1 = 0 \Rightarrow x_1 = \dfrac{1}{9}$

so root is $x = 2 + \dfrac{1}{9}\varepsilon + O(\varepsilon^2)$