Question
Solve $x^2 - 2.01x + 1 = 0$

(i) Exactly

(ii) Using a perturbation series to an accuracy of 0.0001.

Answer

(i)

$$\begin{aligned} x^2 - 2.01x + 1 = 0 \Rightarrow x &= \frac{2.01 \pm \sqrt{4.0401 - 4}}{2} \\ &= \frac{(2.10 \pm 0.200249844)}{2} \\ &= 1.105124922 \cdots \\ &\quad \underline{or} 0.904875078 \cdots \end{aligned}$$

(ii) Roots not distinct at $\varepsilon = 0$, so try perturbation on $\underbrace{x^2 - 2x + 1}_{(x-1)^2 = 0} = 0$

$$x^2 - (2 + \varepsilon)x + 1 = 0 \text{ where } \varepsilon = 0.01$$

Ansatz: $x = x_0 + x_1\varepsilon^{\frac{1}{2}} + x_2\varepsilon + O(\epsilon^{\frac{3}{2}})$

since 2 coincident roots at $\varepsilon = 0$. Check notes and see.

Substitute:

$$[x_0 + x_1\varepsilon^{\frac{1}{2}} + x_2\varepsilon + O(\epsilon^{\frac{3}{2}})]^2 - (2 + \varepsilon)[x_0 + x_1\varepsilon^{\frac{1}{2}} + x_2\varepsilon + O(\varepsilon^{\frac{3}{2}})] + 1 = 0$$

Expand:

$x_0^2 + x_1^2\varepsilon + x_2^2\varepsilon^2 + O(\varepsilon^3)$
probably ok as $\varepsilon^2 = 0.\underline{0001}$ so truncate at O(ε^2)
and hope that implied constant is < 1

$$\begin{aligned} &+ \quad 2x_0x_1\varepsilon^{\frac{1}{2}} + 2x_0x_2\varepsilon + 2x_0x_3\varepsilon^{\frac{3}{2}} + O(\varepsilon^{\frac{5}{2}} \\ &+ \quad 2x_1x_2\varepsilon^{\frac{3}{2}} + 2x_1x_3\varepsilon^{\frac{5}{2}} + O(\varepsilon^{\frac{5}{2}} \\ &+ \quad 2x_2x_3\varepsilon^{\frac{5}{2}} + O(\varepsilon^{\frac{7}{2}} \\ &- \quad 2x_0 - 2x_1\varepsilon^{\frac{1}{2}} - 2x_2\varepsilon - 2x_3\varepsilon^{\frac{3}{2}} - 2x_4\varepsilon^2 + O(\varepsilon^5) \\ &- \quad \varepsilon x_0 - x_1\varepsilon^{\frac{3}{2}} - x_2\varepsilon^2 + O(\varepsilon^{\frac{5}{2}}) \\ &+ \quad 1 = 0 \end{aligned}$$

Balance at $O(\varepsilon^n)$:

$O(\varepsilon^0) : x_0^2 - 2x_0 + 1 = 0 \Rightarrow x_0 \underline{= 1 \text{ twice}}$

$O(\varepsilon^{\frac{1}{2}} : 2x_1 x + 0 - 2x_1 = 0 \Rightarrow x_1(\underbrace{x_0 - 1}) = 0 \Rightarrow x_1 =$ anything! What do we do? Carry on and see.

$O(\varepsilon) : -2x_2 + 2x_0 x_2 + x_1^2 - x_0 = 0 \Rightarrow -2x_2 + 2x_2 + x_1^2 - 1 = 0 \Rightarrow \underline{x_1 = \pm 1}$

$O(\varepsilon^{\frac{3}{2}} : -x_1 + x_1 x_2 - 2x_3 + 2x_0 x_3 = 0 \Rightarrow \mp 1 \pm 2x_2 = 0 \Rightarrow \underline{x_2 = +\frac{1}{2}}$

Actually need to go to $O(\varepsilon^2)$

$O(\varepsilon^2) : x_2^2 - 2x_4 - x_2 + 2x_0 x_4 + 2x_1 x_3 = 0 \Rightarrow \frac{1}{4} - \frac{1}{2} \pm 2x_3 = 0 \Rightarrow x_3 = \pm\frac{1}{8}$

Thus we have a perturbation expansion:

$x = 1 \pm \varepsilon^{\frac{1}{2}} + \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^{\frac{3}{2}} + O(\varepsilon^2)$

This turns out to be sufficient as $O(\varepsilon^2) = \underline{0.0001}$ when $\varepsilon = 0.01$

Therefore

$$\begin{aligned} x &\approx 1 \pm (0.01)^{\frac{1}{2}} + \frac{0.01}{2} \pm \frac{1}{8}(0.001)^{\frac{3}{2}} \\ &= 1.10512_{\uparrow}5 \text{ or } 0.904875_{\uparrow} \end{aligned}$$

which is certainly accurate to 0.0001 by comparison with exact result.

Note: Here we had to calculate the x_i at the $O(\varepsilon^{\frac{i+1}{2}})$ level, due to the strange behaviour at $O(\varepsilon^{\frac{1}{2}})$. Not a problem, just an example of how you may have to go to a higher order (and hence do more work) to find coefficients of lower orders.
Nasty question to start with but good practice!