**Question**

Suppose that $X$ has the pmf $f(x) = q^{x-1}p, \ x = 1, 2, 3, \ldots \ 0 < p < 1, \ p+q = 1$. Find the pgf and the mgf of $X$. Find the mean and the variance of $X$ using the pgf.

**Answer**

The mgf is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=1}^{\infty} e^{tX} q^{x-1} p \\ &= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x \\ &= \frac{p}{q} \{qe^t + (qe^t)^2 + (qe^t)^3 + \ldots\} \\ &= \frac{p}{q} \frac{qe^t}{1-qe^t} \text{ if } |qe^t| < 1 \Leftrightarrow e^t < \frac{1}{q} \Leftrightarrow t < -\log(1-p) \\ &= \frac{pe^t}{1-qe^t} \text{ if } t < -\log(1-p). \end{aligned}$$

We can directly obtain the pgf similarly.

$H(t) = E(t^X) = \dfrac{pt}{1-qt} \quad \text{if } t < \dfrac{1}{q}$

$\dfrac{dH(t)}{dt} = p\dfrac{1-qt+qt}{(1-qt)^2} = \dfrac{p}{(1-qt)^2}$

$\dfrac{d^2H(t)}{dt^2} = \dfrac{2pq}{(1-qt)^3}$

Therefore $E(x) = \left.\dfrac{dH(t)}{dt}\right|_{t=1} = \dfrac{p}{(1-q)^2} = \dfrac{p}{p^2} = \dfrac{1}{p}$

$E\{X(X-1)\} = \left.\dfrac{d^2H(t)}{dt^2}\right|_{t=1} = \dfrac{2pq}{(1-q)^3} = \dfrac{2pq}{p^3} = \dfrac{2q}{p^2}$

Therefore $E(X^2) - E(X) = \dfrac{2q}{p^2} \Rightarrow E(X^2) = \dfrac{2q}{p^2} + \dfrac{1}{p}$.

Therefore

$$\begin{aligned} var(X) &= E(X^2) - \{E(X)\}^2 \\ &= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} \\ &= \frac{2q+p-1}{p^2} \end{aligned}$$

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\begin{aligned}
&= \frac{q+q+p-1}{p^2} \\
&= \frac{q}{p^2}
\end{aligned}
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