QUESTION

(a) Consider the exponential distribution

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

(i) Prove that the mean of the exponential distribution is λ^{-1}.

(ii) Write down an expression which demonstrates that a probability density function has the no-memory property.

(iii) Prove that the exponential distribution has the no-memory property.

(iv) The inter-arrival time of the school buses is believed to be exponentially distributed with a mean of 20 minutes. You have been waiting for the bus for 30 minutes; what is the probability that you have to wait for more than one hour at the end?

(b) Using the mixed congruential generator

$$x_{n+1} = (7x_n + 11) \bmod 31$$

and seed $x_0 = 9$, generate a stream of five random numbers in the interval $[0, 30]$. Use these to generate five random numbers in the interval $[0, 1)$, to three decimal place accuracy.

(c) By using the inverse transformation method, show that $-\frac{1}{\lambda}\ln(U)$ is exponentially distributed with mean λ^{-1}. Here, U is a continuous random variable uniformly distributed over $(0, 1)$.

ANSWER

(a) **(i)**

$$\begin{aligned}
E(x) &= \int_0^\infty t\lambda e^{-\lambda t}\,dt \\
&= \int_0^\infty te^{-\lambda t} d(\lambda t) \\
&= \int_0^\infty -td(e^{-\lambda t}) \\
&= \int_0^\infty e^{-\lambda t}\,dt - te^{-\lambda t}|_0^\infty \\
&= \int_0^\infty e^{-\lambda t}\,dt - 0 \\
&= \frac{1}{\lambda}
\end{aligned}$$

(ii) A probability distribution $f(x)$ is said to have the no-memory property if

$$\text{Prob}(x > t + h | x \geq t) = \text{Prob}(x > h).$$

(iii) For the exponential distribution function, it is clear that

$$\text{Prob}(x > h) = \int_h^\infty \lambda e^{-\lambda t}\, dt = e^{-\lambda h}.$$

We note that

$$\text{Prob}(x > h + t | x \geq t) = \frac{\text{Prob}(x > h + t \text{ and } x \geq t)}{\text{Prob}(x \geq t)}.$$

Therefore we have

$$\text{Prob}(x > h + t \text{ and } x \geq t) = e^{-\lambda(t+h)} \text{ and } \text{Prob}(x \geq t) = e^{-\lambda t}.$$

Hence

$$\text{Prob}(x > t + h | x \geq t) = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = \text{Prob}(x > h).$$

(iv) By the no memory property, the probability is given by

$$\int_3 0\infty \frac{1}{20} e^{-\frac{t}{20}}\, dt = e^{-\frac{3}{2}}.$$

(b) $x_0 = 9,\ x_1 = 12,\ x_2 = 2,\ x_3 = 25,\ x_4 = 0,\ x_5 = 11$ therefore $U_1 = 0.387,\ U_2 = 0.065,\ U_3 = 0.806,\ U_4 = 0,\ U_5 = 0.355.$

(c) We note that cumulative probability distribution of the Exponential distribution is

$$F(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 10e^{-\lambda x}.$$

By using the inverse transform method we have

$$x = F^{-1}(U) = -\frac{1}{\lambda} \log(1 - U)$$

Here U is the continuous random variable uniformly distributed over $(0, 1)$. Since U and $1 - U$ have the same probability distribution,

$$-\frac{1}{\lambda}\log U$$

is also exponentially distributed with mean λ^{-1}.