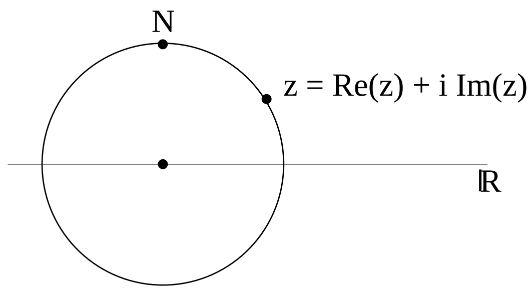
Question

Consider the stereographic projection map ξ from $\mathbf{S}^1$ to $\mathbf{R} \cup \{\infty\}$, as defined in class. Determine the images of the vertices of the regular pentagon with vertices $\exp\left(\frac{2\pi i k}{n}\right)$ for $0 \leq k \leq 4$.

More generally, for each $n \geq 5$, determine the images under ξ of the vertices of a regular n-gon whose vertices lie on $\mathbf{S}^1$ (and where one of the vertices is at 1).

Answer



$\xi(z)$: intersection of line through $i = N$ and z with $\mathbf{R}$.

line through i and z: slope $m = \dfrac{\mathrm{Im}(z) - 1}{\mathrm{Re}(z)}$

equation:

$$y - 1 = m(x - 0)$$
$$y - 1 = \frac{\mathrm{Im(z)} - 1}{\mathrm{Re}(z)} x$$

Set $y = p$, to get $x = \dfrac{\mathrm{Re}(z)}{1 - \mathrm{Im}(z)}$.

So, $\xi(z) = \dfrac{\mathrm{Re}(z)}{1 - \mathrm{Im}(z)}$ $(\xi(n) = \infty)$.

The vertices of the regular n-gon are

$$\exp\left(\frac{2\pi \mathrm{i}}{\mathrm{n}}\mathrm{k}\right) \quad 0 \leq \mathrm{k} < \mathrm{n}$$

So, $\xi\left(\exp\left(\frac{2\pi \mathrm{i}}{\mathrm{n}}\mathrm{k}\right)\right) = \dfrac{\mathrm{Re}\left(\frac{2\pi \mathrm{i}}{\mathrm{n}}\mathrm{k}\right)}{1 - \mathrm{Im}\left(\frac{2\pi \mathrm{i}}{\mathrm{n}}\mathrm{k}\right)} = \dfrac{\cos\left(\frac{2\pi i}{n}k\right)}{1 - \sin\left(\frac{2\pi i}{n}k\right)}$.

(Note that N is a vertex for the regular n-gon for all $n \equiv 0 \pmod 4$)