QUESTION Let $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \in \mathbf{R}^n$. Consider the vector $\lambda\mathbf{u}+\mathbf{v}$ where $\lambda \in \mathbf{R}$, then by positivity
$0 \leq (\lambda\mathbf{u}+\mathbf{v}).(\lambda\mathbf{u}+\mathbf{v}) = (\mathbf{u.u})\lambda^2 + 2(\mathbf{u.v})\lambda + (\mathbf{v.v})$.

(a) Considered as a polynomial in λ, how many real roots does

$p(\lambda) = (\mathbf{u.u}\lambda^2 + 2(\mathbf{u.v})\lambda + (\mathbf{v.v})$

have?

(b) Write down the discriminant of the polynomial $p(\lambda)$.

(c) What does the result in part (a) tell you about the discriminant?

(d) Deduce the Cauchy-Schwarz inequality: $(\mathbf{u.v})^2 \leq (\mathbf{u.u})(\mathbf{v.v})$.

(e) Prove the triangle inequality in $\mathbf{R}^n$: $|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$

[Hint: Rewrite in terms of the inner product: $|u| = \sqrt{(\mathbf{u.u})}$, etc.]

(f) Prove that in $\mathbf{R}^n$: $|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}\text{-}\mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

ANSWER

(a) Since $p(\lambda) \geq 0$ for all λ, there is at most one real root.

(b) $\Delta = 4(\mathbf{u.v})^2 - 4(\mathbf{u.u})(\mathbf{v.v})$.

(c) Since $p(\lambda)$ has at most one real root, $\Delta \leq 0$.

(d) Divide $\Delta \leq 0$ by 4.

(e) In terms of inner products the desired result can be written

$$\begin{aligned}
& \sqrt{\{(\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v})\}} \leq \sqrt{(\mathbf{u.u})} + \sqrt{(\mathbf{v.v})} \\
\Leftrightarrow\quad & (\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v}) \leq \mathbf{u.u} + 2\sqrt{\{(\mathbf{u.u})(\mathbf{v.v})\}} + (\mathbf{v.v}) \\
\Leftrightarrow\quad & \mathbf{u.u} + 2\mathbf{u.v} + \mathbf{v.v} \leq \mathbf{u.u} + 2\sqrt{\{(\mathbf{u.u})(\mathbf{v.v})\}} + (\mathbf{v.v}) \\
\Leftrightarrow\quad & \mathbf{u.v} \leq \sqrt{\{(\mathbf{u.u})(\mathbf{v.v})\}} \\
\Leftarrow\quad & (\mathbf{u.v})^2 \leq (\mathbf{u.u})(\mathbf{v.v})
\end{aligned}$$

but this is just the Cauchy-Schwarz inequality.

(f)

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}\text{-}\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v}) + (\mathbf{u}\text{-}\mathbf{v}).(\mathbf{u}\text{-}\mathbf{v}) \\
&= \mathbf{u.u} + 2\mathbf{u.v} + \mathbf{v.v} + \mathbf{u.u} - 2\mathbf{u.v} + \mathbf{v.v} \\
&= 2(\mathbf{u.u} + \mathbf{v.v}) \\
&= 2(|\mathbf{u}|^2 + |\mathbf{v}|^2).
\end{aligned}$$