Question

(a) Solve the differential equation

$$\frac{dy}{dx} = e^x \cos^2 y, \text{ where } y = \frac{\pi}{4} \text{ when } x = 0.$$

(b) Solve the differential equation

$$x\frac{dy}{dx} + y = \sec x, \text{ where } y = 1 \text{ when } x = 0.$$

Hint: All the integrals you need are in the table at the end of the exam paper.

Answer

(a) $\sec y \frac{dy}{dx} - e^x \cos y = 0$

rewrite as

$$\frac{dy}{dx} = e^x \frac{\cos y}{\sec y} = e^x \cos^2 y$$

This is variables separable

$$\Rightarrow \int \frac{dy}{\cos^2 y} = \int e^x dx$$

$$\Rightarrow \int \sec^2 y \, dy = e^x + c'$$

$$\Rightarrow \tan y = e^x + c$$

Now $y = \frac{\pi}{4}$ when $x = 0$

Therefore $\tan \frac{\pi}{4} = e^0 + c$

$$\Rightarrow 1 = 1 + c$$

$$\Rightarrow c = 0$$

Therefore $\underline{\tan y = e^x}$

or $\underline{y = \arctan(e^x)}$

on some restricted range of y

(b) Rewrite as

$\frac{dy}{dx} + \frac{y}{x} = \frac{\sec x}{x}$ cf $\frac{dy}{dx} + P(x)y = Q(x)$

It's linear first order $\Rightarrow$ integrating factor or exact

$$\begin{aligned} I.F. &= e^{\int \frac{1}{x} dx} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad x\frac{dy}{dx} + x\frac{y}{x} &= x\frac{\sec x}{x} \\ \text{Therefore} \quad \frac{d}{dx}(xy) &= \sec x \\ \Rightarrow \quad xy &= \int \sec x dx \\ &= \ln|\sec x + \tan x| + c \\ \Rightarrow \quad y &= \frac{1}{x}(\ln|\sec x + \tan x| + c) \end{aligned}$$

Now $y = 1$ when $x = 0$ so go back one line.

$$\begin{aligned} 0 \times 1 &= \ln|\sec 0 + \tan 0| + c \\ \Rightarrow 0 &= \ln(1) + c \\ \Rightarrow c &= 0 \end{aligned}$$

Therefore $\underline{y = \frac{\ln|\sec x + \tan x|}{x}}$