Question

Prove that if $x_n \to x$ as $n \to \infty$, then $\frac{x_1+\cdots+x_n}{n} \to x$ as $n \to \infty$.

Answer

Since $\lim_{n\to\infty} x_n = x$, we have that for each $\varepsilon > 0$, there exists M so that $|x_n - x| < \frac{1}{3}\varepsilon$ for $n > M$. For any $m > 0$ and $n > M$, we now have that

$$\begin{aligned} |x_{n+1} + \cdots + x_{n+m} - mx| &= |x_{n+1} - x + \cdots + x_{n+m} - x| \\ &\leq |x_{n+1} - x| + \cdots + |x_{n+m} - x| \\ &\leq m\frac{1}{3}\varepsilon. \end{aligned}$$

Dividing by $n + m$, we obtain that

$$\left|\frac{1}{n+m}(x_{n+1} + \cdots + x_{n+m}) - \frac{m}{n+m}x\right| \leq \frac{m}{n+m}\frac{1}{3}\varepsilon < \frac{1}{3}\varepsilon$$

(since $\frac{m}{n+m} < 1$). Viewing n as fixed for the moment, choose m so that both $|\frac{m}{n+m}x - x| < \frac{1}{3}\varepsilon$ (which we can do since $\lim_{m\to\infty} \frac{m}{n+m} = 1$ for n fixed) and $\frac{1}{n+m}|x_1 + x_2 + \cdots + x_n| < \frac{1}{3}\varepsilon$ (which we can do since $x_1 + x_2 + \cdots + x_n$ is a constant when n is fixed). Then,

$$\begin{aligned} &\left|\frac{1}{n+m}(x_1 + \cdots + x_{n+m}) - x\right| \\ &\quad = \left|\frac{1}{n+m}(x_1 + \cdots + x_n) + \frac{1}{n+m}(x_{n+1} + \cdots + x_{n+m}) - \frac{m}{n+m}x + \frac{m}{n+m}x - x\right| \\ &\quad \leq \left|\frac{1}{n+m}(x_1 + \cdots + x_n)\right| + \left|\frac{1}{n+m}(x_{n+1} + \cdots + x_{n+m}) - \frac{m}{n+m}x\right| + \left|\frac{m}{n+m}x - x\right| \\ &\quad \leq \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

for all $m > 0$. Since this is true for all $n > M$ and all $m > 0$, we have that $\left|\frac{1}{p}(x_1 + \cdots + x_p) - x\right| < \varepsilon$ for all $p > M$, as desired.