Question
Using the definition of limit, prove that $\lim_{n\to\infty} \frac{1+2\cdot 10^n}{5+3\cdot 10^n} = \frac{2}{3}$. For what value of M do we have that $|\frac{1+2\cdot 10^n}{5+3\cdot 10^n} - \frac{2}{3}| < 10^{-3}$ for all $n > M$?

Answer
Set $a_n = \frac{1+2\cdot 10^n}{5+3\cdot 10^n}$ and $L = \frac{2}{3}$. For each choice of $\varepsilon > 0$, we need to show that there exists M so that $|a_n - L| < \varepsilon$ for all $n > M$.

Calculating, we see that

$$|a_n - L| = \left|\frac{1+2\cdot 10^n}{5+3\cdot 10^n} - \frac{2}{3}\right| = \left|\frac{3+6\cdot 10^n - (10+6\cdot 10^n)}{3(5+3\cdot 10^n)}\right| = \left|\frac{7}{15+9\cdot 10^n}\right|.$$

Hence, for a given value of $\varepsilon > 0$, we want to find M so that $\left|\frac{7}{15+9\cdot 10^n}\right| < \varepsilon$ for $n > M$. So, we solve for n in terms of ε. First, note that $\frac{7}{15+9\cdot 10^n} > 0$ for all positive integers n. So, we need only solve $\frac{7}{15+9\cdot 10^n} < \varepsilon$ for n.

So, $\frac{7}{\varepsilon} < 15+9\cdot 10^n$, and so $-15+\frac{7}{\varepsilon} < 9\cdot 10^n$, and so $-\frac{15}{9}+\frac{7}{9\varepsilon} < 10^n$. Performing a final bit of simplification, we get $\frac{-15\varepsilon+7}{9\varepsilon} < 10^n$. If the numerator is positive, that is if $\varepsilon < \frac{7}{15}$, we can solve for n by taking $\log_{10}$ of both sides. If on the other hand the numerator is negative, then any positive integer will do. So, set

$$M = \begin{cases} 1 & \text{if } \varepsilon \geq \frac{7}{15}; \\ \log_{10}\left(\frac{-15\varepsilon+7}{9\varepsilon}\right) & \text{otherwise} \end{cases}$$

To get a specific value of M so that $|a_n - L| < 10^{-3}$ for $n > M$, we substitute $\varepsilon = 10^{-3}$ into the above equation to get that $n > \log_{10}\left(\frac{-15\cdot 10^{-3}+7}{9\cdot 10^{-3}}\right) \approx 2.8899$. So, we can take $M = 3$.