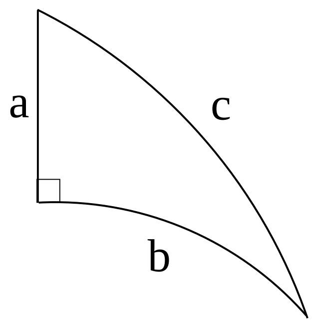
Question
State and prove the hyperbolic Pythagorean theorem, relating the hyperbolic lengths of the three sides of a hyperbolic right triangle.

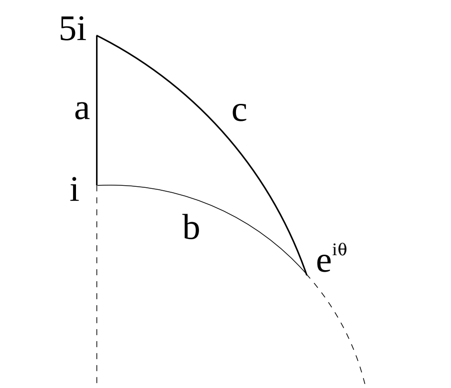
Use this hyperbolic Pythagorean theorem to rework Exercise 3, Sheet 8.

Answer



$$\begin{aligned}\cosh(c) = {} & \cosh(a)\cosh(b) \\ & -\sinh(a)\sinh(b)\cos(\gamma) \\ = {} & \cosh(a)\cosh(b) \\ & \underline{\text{from lcI}}\end{aligned}$$

<u>Exercise 3, sheet 8</u>



$a = \ln(5)$

$$\begin{aligned}\cosh(a) &= \frac{1}{2}\left(e^{\ln(5)} + e^{-\ln(5)}\right) \\ &= \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}\end{aligned}$$

$$\begin{aligned}b = \int_{\theta}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt &= -\ln|\csc(\theta) - \cot(\theta)| \\ &= \ln\left|\frac{\sin(\theta)}{1-\cos(\theta)}\right|\end{aligned}$$

$$\cosh(b) = \frac{1}{2}(e^b + e^{-b})$$

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{\sin(\theta)}{1-\cos(\theta)} + \frac{1-\cos(\theta)}{\sin(\theta)}\right) \\
&= \frac{1}{2}\left(\frac{\sin^2(\theta) + (1-\cos(\theta)^2}{\sin(\theta)(1-\cos(\theta))}\right) \\
&= \frac{\sin^2(\theta) + 1 - 2\cos(\theta) + \cos^2(\theta)}{2\sin(\theta)(1-\cos(\theta))} \\
&= \frac{1}{\sin(\theta)} \\
&= \underline{\csc(\theta)}
\end{aligned}$$

so $\cosh(c) = \cosh(a)\cos(b) = \frac{13}{5}\csc(\theta)$

$$e^c + e^{-c} = \frac{26}{5}\csc(\theta)$$

$$e^{2c} - \frac{26}{5}\csc(\theta)e^c + 1 = 0$$

$$\begin{aligned}
e^c &= \frac{1}{2}\left[\frac{26}{5}\csc(\theta) + \sqrt{\left(\frac{26}{5}\right)^2\csc^2(\theta) - 4}\right] \\
&= \frac{13}{5}\csc(\theta) + \sqrt{\left(\frac{13}{5}\right)^2\csc^2(\theta) - 1} \\
c &= \underline{\ln\left(\frac{13}{5}\csc(\theta) + \sqrt{\left(\frac{13}{5}\right)^2\csc^2(\theta) - 1}\right)}
\end{aligned}$$