Question
Let D_s be the hyperbolic disc in the Poincaré disc $\mathbf{D}$ with hyperbolic radius s, and let C_s be the hyperbolic circle with hyperbolic radius s that bounds D_s. Describe the behavior of the quotient

$$q(s) = \frac{\text{length}_{\mathbf{D}}(C_s)}{\text{area}_{\mathbf{D}}(D_s)}$$

as $s \to 0$ and as $s \to \infty$.

Compare the behavior of q with the analogous quantity calculated using a Euclidean disc and a Euclidean circle.

Answer
We know from exercise sheet 8 that $\text{length}_{\mathbf{D}}(\mathbf{C}_s) = 2\pi \sinh(s)$.
To calculate $\text{area}_{\mathbf{D}}(\mathbf{D}_s)$:
Recall that the euclidean radius of $\mathbf{D}_s$ is $R = \tanh(\frac{1}{2}s)$, and so

$$\begin{aligned}
area_{\mathbf{D}}(\mathbf{D}_s) &= \int_0^{2\pi}\int_0^R \frac{4}{(1-|z|^2)^2}\, dx\, dy \\
&= \int_0^{2\pi}\int_0^R \frac{4r\, dr\, d\theta}{(1-r^2)^2} \\
&= 8\pi \int_0^R \frac{r\, dr}{(1-r^2)^2} \\
&= 8\pi\, \frac{1}{2}(1-r^2)\Big|_0^R \\
&= 8\pi \left(\frac{1}{2(1-R^2)} - \frac{1}{2} \right) \\
&= \frac{4\pi}{1-R^2} - 4\pi \\
&= \frac{4\pi(1-1+R^2)}{1-R^2} \\
&= \frac{4\pi \tanh^2(\frac{1}{2}s)}{1-\tanh^2(\frac{1}{2}s)} \\
&= 4\pi \sinh^2(\frac{1}{2}s)
\end{aligned}$$

and so

$$\begin{aligned} q(s) = \frac{\text{length}_{\mathbf{D}}(\mathrm{C_s})}{\text{area}_{\mathbf{D}}(\mathrm{D_s})} &= \frac{2\pi \sinh(s)}{4\pi \sinh^2(\frac{1}{2}s)} \\ &= \frac{4\pi \sinh(\frac{1}{2}s)\cosh(\frac{1}{2}s)}{4\pi \sinh^2(\frac{1}{2}s)} \\ &= \frac{\cosh(\frac{1}{2}s)}{\sinh(\frac{1}{2}2)} \\ &= \frac{e^{\frac{1}{2}s} + e^{\frac{-1}{2}s}}{e^{\frac{1}{2}s} - e^{\frac{-1}{2}s}} \\ &= \frac{e^s + 1}{e^s - 1} \end{aligned}$$

as $s \to \infty,\ q(s) \to 1$.
as $s \to 0^+,\ q(s) \to \infty$ (since numerator $\to 2$ and denominator $\to 0$.)

The analagous euclidean quantity is

$$q_E(s) = \frac{\text{length}_{\mathbf{D}}(\mathrm{C_s})}{\text{area}_{\mathbf{D}}(\mathrm{D_s})} = \frac{2\pi s}{\pi s^2} = \frac{2}{s}$$

as $s \to \infty,\ q_E(s) \to 0$ (different from $q(s)$).
as $s \to 0^+,\ q_E(s) \to \infty$ (as $q(s)$)