QUESTION

(a) Sketch the region defined by the inequalities $x^2 + y^2 \leq x, 0 \leq z \leq 3$. If the region is occupied by a solid whose density at the point (x, y, z) is $x^2 + y^2 + z$ calculate its total mass by means of an appropriate triple integral.

(b) A cube having side length 2 has density at a point given by twice the square of its distance from the center of the cube. Find the mass of the cube.

ANSWER

(a) DIAGRAM

$$\begin{aligned}\text{Mass} &= \iiint_R x^2 + y^2 + z\, dV \\ &= \iiint_R (\rho^2 + z)\rho\, dz d\rho d\phi\end{aligned}$$

in cylindrical coordinates.

$$R = \{(\rho, \phi, z) | 0 \leq \rho \leq \sqrt{3}, 0 \leq \phi \leq 2\pi, \rho^2 \leq z \leq 3\}$$

so

$$\begin{aligned}\text{Mass} &= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\sqrt{3}} \int_{z=\rho^2}^{3} (\rho^3 + \rho z)\, dz d\rho d\phi \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left[\rho^3 z + \rho \frac{z^2}{2}\right]_{\rho^2}^{3} ds d\phi \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(3\rho^3 + \frac{9}{2}\rho - 3\frac{\rho^5}{2}\right) d\rho d\phi \\ &= \int_0^{2\pi} \left[\frac{3}{4}\rho^4 + \frac{9\rho^2}{4} - \frac{\rho^6}{4}\right]_0^{\sqrt{3}} d\phi \\ &= \int_0^{2\pi} \left(\frac{27}{4} + \frac{27}{4} - \frac{27}{4}\right) d\phi \\ &= \frac{27}{2}\pi\end{aligned}$$

(b) The cube consists of eight identical octants so its mass is given by

$$
\begin{aligned}
& 8\int_0^1\int_0^1\int_0^1 2(x^2+y^2+z^2)\,dxdydz \\
= \;& 8\int_0^1\int_0^1 \left[\frac{2x^3}{3}+2(y^2+z^2)x\right]_0^1 dydz \\
= \;& 8\int_0^1\int_0^1 \left(\frac{2}{3}+2y^2+2z^2\right) dydz \\
= \;& 8\int_0^1 \left[\left(\frac{2}{3}+2z^2\right)y+2\frac{y^3}{3}\right]_0^1 dz \\
= \;& 8\int_0^1 \left(\frac{2}{3}+2z^2+\frac{2}{3}\right) dz \\
= \;& 8\left[\frac{4z}{3}+2\frac{z^3}{3}\right]_0^1 \\
= \;& 16
\end{aligned}
$$