Question

(i) Show that if,

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx, \quad n \geq 1,$$

then by integrating by parts twice,

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

Hint: remember you want to try to reduce the power of n.

(ii) Evaluate I_0 and hence calculate I_6.

(iii) Given that when $n = \dfrac{1}{2}$, $I_{\frac{1}{2}} = 0.977451...$, calculate $I_{\frac{5}{2}}$ to six decimal places.

Answer

(i) $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x \, dx \quad n \geq 1$

$$u = x^n \qquad \frac{dv}{dx} = \sin x$$
$$du = nx^{n-1} \qquad v = -\cos x$$

$\Rightarrow$

$$\begin{aligned} I_n &= [-x^n \cos x]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x \, dx \\ &= -\left(\frac{\pi}{2}\right)^n \cos\left(\frac{\pi}{2}\right) - 0^n \cos 0 + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x \, dx \\ &= n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x \, dx \end{aligned}$$

$$u = x^{n-1} \qquad \frac{dv}{dx} = \cos x$$
$$du = (n-1)x^{n-2} \qquad v = \sin x$$

$\Rightarrow$

$$\begin{aligned} I_n &= n\left\{[x^{n-1}\sin x]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x\,dx\right\} \\ &= n\left(\left(\frac{\pi}{2}\right)^{n-1}\sin\frac{\pi}{2} - 0^{n-1}\sin 0\right] - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx \end{aligned}$$

Therefore $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-1}$

(ii) $I_0 = \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$

$I_6 = 6\left(\frac{\pi}{2}\right)^5 - 6\times 5\times I_4$

$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 4\times 3\times I_2$

$I_2 = 2\left(\frac{\pi}{2}\right)^1 - 2\times 1\times I_0 = 2\frac{\pi}{2} - 2\times 1 = \pi - 2$ Back substitute

$I_4 = \frac{4}{8}\pi^3 - 12(\pi - 2)$

$I_6 = \frac{6}{32}\pi^5 - 30\left(\frac{\pi^3}{2} - 12(\pi-2)\right)$

$\frac{3\pi^5}{16} - 15\pi^3 + 360\pi - 720 = 3.257896...$

(iii) $I_{\frac{5}{2}} = \frac{5}{2}\left(\frac{\pi}{2}\right)^{\frac{3}{2}} - \frac{5}{2}\cdot\frac{3}{2}\cdot I_{\frac{1}{2}} = 1.256311858...$