Question
By applying the Period-Doubling Theorem to $g_a^2 = [ax(1-x)]^2$ show that a 4-cycle is created as a increases through $1+\sqrt{6}$.

Answer If the 2-cycle is $\{p, q\}$ as in question 6, then

$$\begin{aligned}(g_a^4)' &= {g_a}'(g_a^3(x)){g_a}'(g_a^2(x)){g_a}'(g_a(x))g_a(x)\\ &= a^4(1-2g_a^3(x)(1-2g_a^2(x))(1-2g_a(x))(1-2x)\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial g_a}{\partial a}(p) &= p(1-p) = \frac{1}{a}q;\\ \frac{\partial {g_a}^2}{\partial a}(p) &= \frac{1}{a}p + a(1-2q)\frac{1}{a}q;\\ \frac{\partial {g_a}^3}{\partial a}(p) &= \frac{1}{a}q + a(1-2p)\left[\frac{1}{a}p + a(1-2q)\frac{1}{a}q\right].\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial a}(g_a^4)'(p) &= 4a^3(1-2p)^2(1-2q)^2\\ &- 2a^4\left(\frac{1}{a}q + a(1-2p)\left[\frac{1}{a}p + a(1-2q)\frac{1}{a}q\right]\right)(1-2p)^2(1-2q)\\ &- 2a^4(1-2q)\left(\frac{1}{a}p + a(1-2q)\frac{1}{a}q\right)(1-2q)(1-2p)\\ &- 2a^4(1-2q)(1-2p)\frac{1}{a}q(1-2p)\end{aligned}$$

When $a = 1+\sqrt{6}$ we have $(g_a^2)'(p) = a^2(1-2p)(1-2q) = -1$ so we find the above simplifies to

$$\begin{aligned}\left.\frac{\partial}{\partial a}(g_a^4)'(p)\right|_{a=1+\sqrt{6}} &= \frac{4}{a} + 2a^2p(1-2p)^2 + 2a^2(1-2q)\left(\frac{1}{a}p + (1-2q)q\right) + 2aq(1-2p)\\ &= \frac{4}{a} + 2a^2(p(1-2p)^2 + q(1-2q)^2) + 2a(p(1-2q) + q(1-2p)).\end{aligned}$$

Now using $p + q = \frac{1}{a} + 1$ (=b, say) and $pq = \frac{b}{a}$ we find the above

$$\begin{aligned}&= \frac{4}{a} + 2a^2b\left(1 - \frac{8}{a^2}\right) + 2ab\Big(1 - \frac{4}{a}\\ &= \frac{4}{a} + 2b(a^2 + a - 12) > 0\end{aligned}$$

because $a_a^2 - 12 = (1+\sqrt{6})^2 + (1+\sqrt{6}) - 12 = 3\sqrt{6} - 4 > 0$.
Hence the bifurcation from a 2-cycle to a 4-cycle is supercritical.