Question

Show that $f : \mathbf{R} \longrightarrow \mathbf{R} : x + x^3$ has a repelling fixed point at $x = 0$ (although $f'(0) = 1$). What about $x \mapsto x - x^3$? What is the behaviour of $x \mapsto -x + x^3,\ x \mapsto -x - x^3$ near the origin?

Answer

$\underline{f(x) = x + x^3}$ $\Rightarrow f(x) > x$ or $< x$ according as $x > 0$ or < 0. Thus $f^n(x)$ tends monotonically away from 0, and in fact $f^n(x) \to \pm\infty$ (else $f^n(x) \to l$ for some finite l, which then has $f(l) = l$: not the case for $l \neq 0$).

$\underline{f(x) = x - x^3}$: attracting fixed pt. at 0, since $0 < x < 1 \Rightarrow 0 < f(x) < x < 1$ $(-1 < x < 0 \Rightarrow -1 < x < f(x) < 0)$ so $f(x) \to$ *limit* m which has $f(m) = m$ so $m = 0$.

$\underline{f(x) = -x + x^3}$:

$$\begin{aligned} \text{here } 0 < x < 1 \quad &\Rightarrow \quad -1 < -x < f(x) < 0 \\ \text{and } -1 < x < 0 \quad &\Rightarrow \quad 0 < f(x) < -x < 1 \end{aligned}$$

so $|x| < 1 \Rightarrow |f(x)| < |x|$ so $|f^n(x)| \to l$ and $|f(l)| = |l|$ so $l = 0$: attracting. Likewise $\underline{f(x) = -x - x^3}$: origin is repelling. (Last 2 cases with oscillation.)