Question

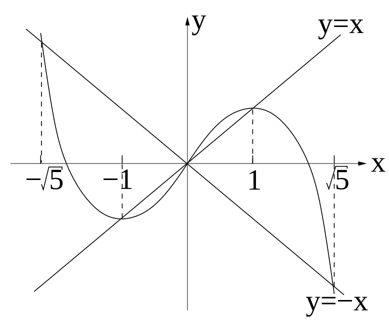
Let $f(x) = \frac{1}{2}(3x - x^3)$. Show that the 2-cycle $\left\{\pm\sqrt{5}\right\}$ is repelling. Solve the inequality $|f(x)| > |x|$ and also $|f(x)| < |x|$ to show there are no periodic points apart from $\left\{\pm\sqrt{5}\right\}$ and the fixed points $\{0, \pm 1\}$.

Answer

$f'(x) = \dfrac{3}{2}(1 - x^2)$, so $f'(\pm\sqrt{5}) = \dfrac{3}{2}(-4) = 6$.

Hence $(f^2)'(\sqrt{5}) = f(\sqrt{5})(-\sqrt{5}) = 36 > 1$ so the 2-cycle $\{\sqrt{5}, -\sqrt{5}\}$ is repelling.

From the graph of f (or considering $f(x) > x, f(x) < -x$ etc.) we see that $|f(x)| > |x|$ when $0 < |x| < 1$ or $|x| > \sqrt{5}$.



Graphical iteration shows $x \mapsto 1$ or -1 in the first case, and $x \mapsto \infty$ in the second. If $1 < |x| < \sqrt{5}$ then $|f(x)| < |x|$, so either eventually $|f^m(x)| < 1$ (so $f^n(x) \to 1$ or -1) or $|f^m(x)| \to$ *limit* $l \geq 1$: in this case $|f(l)| = |l|$ so again $l \pm 1$. So no more periodic orbits.