Question

Show that the following can be made exact using an integrating factor that only depends on x and hence find the solution in each case.

1. $(x^2 + 2y)dx - x\,dy = 0 \qquad (*)$
2. $(xe^x + x\ln y + y)dx + \left(\dfrac{x^2}{y} + x\ln x + x\sin y\right)dy = 0$

Answer

a) $p = x^2 + 2y, \quad q = -x, \quad \Rightarrow \dfrac{\partial p}{\partial y} = 2, \quad \dfrac{\partial q}{\partial x} = -1 \Rightarrow$ not exact.

Now, either guess a possible multiple to make the equation exact or multiply by $H(x) \Rightarrow p = (x^2 + 2y)H, \quad q = -xH.$

$\Rightarrow \dfrac{\partial p}{\partial y} = 2H, \quad \dfrac{\partial q}{\partial x} = -H - x\dfrac{dH}{dx}$ so for it to be exact we need

$2H = -H - x\dfrac{dH}{dx} \Rightarrow 3H = -x\dfrac{dH}{dx} \Rightarrow \displaystyle\int \frac{1}{x}dx = \int \frac{1}{3H}dH$

$\ln x + A = \dfrac{1}{3}\ln H \Rightarrow H = (x + A)^{-3}$, we only need one H so let $A = 0 \Rightarrow H = x^{-3}$.

Now the equation is $\left(\dfrac{1}{x} + \dfrac{2y}{x^3}\right)dx - \dfrac{1}{x^2}dy = 0$

$p = \dfrac{1}{x} + \dfrac{2y}{x^3}, \quad q = -\dfrac{1}{x^2}, \quad \dfrac{\partial F}{\partial x} = \dfrac{1}{x} + \dfrac{2y}{x^3}, \quad \dfrac{\partial F}{\partial y} = -\dfrac{1}{x^2}$

$\Rightarrow F = -\dfrac{y}{x^2} + g(x) \Rightarrow \dfrac{\partial F}{\partial x} = \dfrac{2y}{x^3} + \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow y = \ln x + c$

so the solution is $-\dfrac{y}{x^2} + \ln x = A$ or $y = x^2(c + \ln x)$

b) Can do in the same way as 5a) or notice that with

$q = \dfrac{x^2}{y} + x \ln x + x \sin y$ when you evaluate $\dfrac{\partial q}{\partial x}$ you get a term with $\sin y$ and there is no such term in $p(x, y)$ or $\dfrac{\partial p}{\partial y}$. Hence multiply equation by $\dfrac{1}{x}$ to get

$\left(e^x + \ln y + \dfrac{y}{x}\right) dx + \left(\dfrac{x}{y} + \ln x + \sin y\right) dy = 0$ which is exact.

The solution is $e^x + x \ln y + y \ln x - \cos y = A$