QUESTION

Decide whether or not the following quadratic equations have solutions. If there are any solutions, find them.

(i) $3x^2 + 5x + 1 \equiv 0 \bmod 7$.

(ii) $2x^2 + 3x + 6 \equiv 0 \bmod 11$.

ANSWER

(i) A quadratic congruence $ax^2 + bx + c \equiv 0 \bmod p$, where p is an odd prime such that $p \not| a$, has roots if and only iff the discriminant $d = b^2 - 4ac$ is a square mod p.

Here $d \equiv 25 - 4.3.1 \equiv 13 \equiv -1 \bmod 7$. By Theorem 7.1(v), -1 is not a square mod 7, so our equation has no solutions.

(ii) $2x^2 + 3x + 6 \equiv 0 \bmod 11$.

Arguing as in (i), $d \equiv 9 - 2.4.5 \equiv -39 \equiv 5 \bmod 11$. Now $5 \equiv 16 \equiv 4^2 \bmod 11$, so here solutions exist.

By multiplying our original equation through by 8 ($= 4a$) we obtain $(4x + 3)^2 \equiv 9 - 8.6 \equiv 5 \bmod 11$, so on noting that $5 \equiv (\pm 4)^2 \bmod 11$ we see that we have two possible solutions, viz. $4x + 3 \equiv 4 \bmod 11$ and $4x + 3 \equiv -4 \bmod 11$. These simplify to $4x \equiv 1 \bmod 11$ and $4x \equiv -7 \equiv 4 \bmod 11$, giving solutions $x \equiv 3$ and $x \equiv 1 \bmod 11$, respectively.

[Other methods of solution are possible - e.g. you may have spotted 1 as a root, and then factorised the equation as $(x - 1)(2x - 6) \equiv 0 \bmod 11$.]