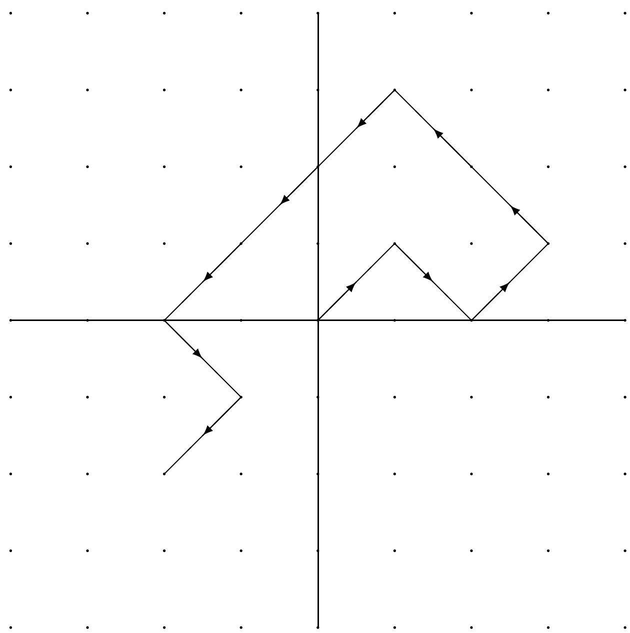
Question

Starting from an origin a particle moves in a plane in two-dimensional simple random walk. At each step it can move one unit north or south (each with probability $\frac{1}{2}$) and, independently, one unit east or west (each with probability $\frac{1}{2}$). After the step is taken, the move is repeated from the new position and so on indefinitely. Sketch one realisation of this process. By regarding the process as the product of two independent simple random walks, calculate the probability that the particle is at the origin after k steps. Hence show that with probability 1 the particle returns infinitely often to the origin.

Answer



The reasoning is very similar to the 1-dimensional case. To return to 0 the particle must do so as a result of a simultaneous return in in respect of movement both east/west and north/south.

$$P(\text{Particle is at 0 after } k \text{ steps}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \left[\begin{pmatrix} 2n \\ n \end{pmatrix}\left(\frac{1}{2}\right)^{2n}\right]^2 & \text{if } k = 2n \end{cases}$$

The expected number of returns to the origin is

$$E = \sum_{k=1}^{\infty} E(R_k) \cdot R_k = \begin{cases} 1 & \text{if particle at 0 after } k \text{ steps} \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} P(R_k = 1) \\
&= \sum_{n=1}^{\infty} \left[\binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right]^2 \\
&\approx \sum_{n=1}^{\infty} \frac{1}{\pi n}
\end{aligned}
$$

using Stirling's approximation. The series diverges, so $E = \infty$. Standard theory gives $P = 1 - \dfrac{1}{1+E}$ so $P = 1$