Question

A population consists of b individuals. During a time interval $(t, t + \delta t]$ each individual in the population has, independently of all the other individuals and of what happened on $[0, t]$, a probability $\mu\delta t + o(\delta t)$ of dying. Thus the population decreases from b to 0. Let $X(t)$ be the size of the population after time t and $p_n(t)$ denote the probability that $X(t) = n$ $(n = 0, 1, \ldots, b)$. Show that:

(i) $P\{X(t + \delta t) = n - 1|X(t) = n\} = n\mu\delta t + o(\delta t)$, and

$P\{X(t + \delta t) = n|X(t) = n\} = 1 - n\mu\delta t + o(\delta t)$ as $\delta t \to 0$

(ii) $p_b'(t) = -b\mu p_b(t)$.

(iii) $p_n'(t) = -p_n(t)n\mu + p_{n+1}(t)(n + 1)\mu$, for $n = 0, 1, \ldots, b - 1$.

(iv) the generating function, $G(z, t)$, of $X(t)$ satisfies the differential equation

$$\frac{\partial G}{\partial t} = \mu(1 - z)\frac{\partial G}{\partial z}$$

(v) $G(z, t) = e^{-\mu tb}(z + e^{\mu t} - 1)^b$.

Hence find the distribution of $X(t)$ and comment on whether or not this result is surprising. The process $\{X(t); t \geq 0\}$ is called a *linear death process.*

Answer

(i) $P(X(t + \delta t) = n - 1|X(t) = n) = P(\text{only 1 out of } n \text{ individuals dies})$

$$\begin{aligned} &= n[\mu\delta t + o(\delta t)][1 - \mu\delta t + o(\delta t)]^{n-1} \\ &= n\mu\delta t + o(\delta t) \quad \text{as } \delta t \to 0 \\ &\quad [P(X(t + \delta t) = n|X(t) = n) = P(\text{no individual dies})] \\ &= [1 - \mu\delta t + o(\delta t)]^n = 1 - n\mu\delta t + o(\delta t) \quad \text{as } \delta t \to 0 \end{aligned}$$

(ii) $P_b(t + \delta t) = P(X(t) = b \text{ and no individual dies in } (t, t + \delta t])$

$= p_b(t)[1 - b\mu\delta t + o(\delta t)]$ by independence (Markov property)

Thus $P_b'(t) = -b\mu P_b(t)$

(iii) for $n = 0, 1, \ldots, b - 1$

$$
\begin{aligned}
p_n(t+\delta t) &= P(X(t)=n \text{ and no deaths in } (t,t+\delta t]) \\
&\quad +P(X(t)=n+1 \text{ and 1 death in } (t,t+\delta t]) \\
&\quad +P(X(t)>n+1 \text{ and } >1 \text{ death in } (t,t+\delta t]) \\
&= p_n(y)[1-n\mu\delta t+o(\delta t)] \\
&\quad +p_{n+1}(t)[(n+1)\mu\delta t+o(\delta t)]+o(\delta t)
\end{aligned}
$$

Thus $p_n'(t) = -n\mu p_n(t) + (n+1)\mu p_{n+1}(t)$

(iv) $G(z,t) = \sum_{n-0}^{\infty} p_n(t) z^n$

$$
\begin{aligned}
\frac{\partial G}{\partial t} &= \sum_{n=0}^{\infty} p_n'(t) z^n \\
&= \sum_{n=0}^{b-1} -n\mu p_n(t) z^n + \sum_{n=0}^{b-1} (n+1)\mu p_{n+1}(t) z^n - b\mu p_b(t) z^b \\
&= -\sum_{n=1}^{b} n\mu p_n(t) z^n + \sum_{n=1}^{b} n\mu p_n(t) z^{n-1} \\
&= \mu(1-z)\frac{\partial G}{\partial z}
\end{aligned}
$$

(v) Let $G(z,t) = e^{-\mu tb}(z+e^{\mu t}-1)^b$

$$
\begin{aligned}
\frac{\partial G}{\partial t} &= -\mu b e^{-\mu tb}(z+e^{\mu t}-1)^b + e^{-\mu tb} b (z+e^{\mu t}-1)^{b-1}\mu e^{\mu t} \\
&= \mu b e^{-\mu tb}(z+e^{\mu t}-1)^{b-1}(e^{\mu t}-(z+e^{\mu t}-1)) \\
&= \mu(1-z) b e^{-\mu tb}(z+e^{\mu t}-1)^{b-1} \\
&= \mu(1-z)\frac{\partial G}{\partial z}
\end{aligned}
$$

Now $p_b(0)=1$ and $p_n(0)=0$ for $n \neq b$ so $G(z,0)$ should be z^b, which is the case.

To find $p_n(t)$ we need to expand $G(z,t)$ as a polynomial in z. The coefficient of z^n is

$$
\begin{aligned}
p_n(t) &= \begin{pmatrix} b \\ n \end{pmatrix} e^{-\mu tb}(e^{\mu t}-1)^{b-n} \\
&= \begin{pmatrix} b \\ n \end{pmatrix} e^{-\mu tn}(1-e^{\mu t})^{b-n}
\end{aligned}
$$

So $X(t) \sim B(b, e^{-\mu t})$ - binomial

$G(z,t) = (e^{-\mu t}z + (1 - e^{-\mu t}))^b$ - binomial p.g.f.

Each of the b members of the population has a probability $e^{-\mu t}$ of surviving longer than time t, independently of the others.

So we have a binomial situation, with survival being a Bernouilli trial.