Question

The Bernoulli-Laplace model of diffusion describes the flow of two incompressible liquids between two containers. It may be described in terms of d white and d black balls distributed between two boxes so that each box contains d balls. At each independent trial one ball is drawn from each box at random and placed in the opposite box so that each box always contains d balls. Suppose X_n denotes the number of white balls in box 1 after the n-th trial. Show that $\{X_n\} \quad (n = 1, 2, \ldots)$ forms a Markov chain and find the 1-step transition probabilities. Show that the stationary distribution for this Markov chain is

$$\pi_k = \binom{d}{k}^2 \pi_0, \quad k = 1, 2, \ldots, d,$$

where

$$\pi_0 = \left[\sum_{k=0}^{d} \binom{d}{k}^2\right]^{-1}, \quad \binom{d}{k} = \frac{d!}{k!(d-k)!}.$$

Answer

X_n has possible states $0, 1, 2, \ldots, d$.
$P(X_{n+1} = k)$ depends only on the number of balls in each box box before the $(n+1)$-th trial i.e. by the value X_n, so we have a Markov chain.
Suppose $X_n = j$

Box 1			Box 2		
	j	W		$d-j$	W
	$d-j$	B		j	B

The possible outcomes from the next trial are:

(i) W from 1 and W from 2 with probability $\left(\frac{j}{d}\right) \cdot \left(\frac{d-j}{d}\right)$ giving $X_{n+1} = j$

(ii) W from 1 and B from 2 with probability $\left(\frac{j}{d}\right) \cdot \left(\frac{j}{d}\right)$ giving $X_{n+1} = j - 1$

(iii) B from 1 and W from 2 with probability $\left(\frac{d-j}{d}\right) \cdot \left(\frac{d-j}{d}\right)$ giving $X_{n+1} = j + 1$

(iv) B from 1 and B from 2 with probability $\left(\frac{d-j}{d}\right) \cdot \left(\frac{j}{d}\right)$ giving $X_{n+1} = j$

So $p_{jj} = 2 \cdot \left(\frac{d-j}{d}\right)\left(\frac{j}{d}\right)$
$p_{j,j+1} = \left(\frac{d-j}{d}\right)^2$
$p_{j,j-1} = \left(\frac{j}{d}\right)^2$
$p_{j,k} = 0$ if $k \neq j-1, j, j+1$
With special cases: -
$j = 0$: only (iii) is possible, with probability 1.
$j = d$: only (ii) is possible, with probability 1.

$$P = \frac{1}{d}\begin{pmatrix} 0 & d^2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2(d-1) & (d-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & 2^2 & 2(d-2)\cdot 2 & (d-2)^2 & & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & & & (d-1)^2 & 2\cdot(d-1) & 1 \\ 0 & 0 & & & & d^2 & 0 \end{pmatrix}$$

The stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_d)$ satisfies $\pi P = \pi$, so

$$\pi_0 = \frac{\pi_1}{d^2} \qquad \pi_d = \frac{\pi_{d-1}}{d^2},$$

and for $j \neq 0$ or d,
$\pi_j = \pi_{j-1}\left(\frac{d-j+1}{d}\right)^2 + \pi_j 2\left(\frac{j}{d}\right)\left(\frac{d-j}{d}\right) + \pi_{j+1}\left(\frac{j+1}{d}\right)^2$
Now $\pi_1 = d^2\pi_0 = \binom{d}{1}^2 \pi_0$
We proceed by induction

$$\begin{aligned} \pi_{j+1} &= \left(\frac{d}{j+1}\right)^2\left(\pi_j - \pi_{j-1}\left(\frac{d-j+1}{d}\right)^2 - \pi_j 2\left(\frac{j}{d}\right)\left(\frac{d-j}{d}\right)\right) \\ &= \left(\frac{d}{j+1}\right)^2\left[\left(\frac{d!}{(d-j)!j!}\right)^2\right. \\ &\qquad -\left(\frac{d!}{(d-j+1)!(j-1)!}\right)\left(\frac{d-j+1}{d}\right)^2 \end{aligned}$$

$$
\begin{aligned}
& - \left(\frac{d!}{(d-j)!j!}\right)^2 \cdot 2\left(\frac{j}{d}\right)\left(\frac{d-j}{d}\right)\Bigg] \pi_0 \\
= \; & \left(\frac{d}{j+1}\right)^2 \left[\left(\frac{d!}{(d-j)!j!}\right)^2 - \frac{1}{d^2}\left(\frac{d!}{(d-j)!(j-1)!}\right)^2\right. \\
& \left. - \left(\frac{d!}{(d-j)!j!}\right)^2 \cdot 2\frac{j}{d}\frac{d-j}{d}\right] \pi_0 \\
= \; & \pi_0 \left(\frac{d}{j+1}\right)^2 \left(\frac{d!}{(d-j)!j!}\right)^2 \left[1 - \frac{j^2}{d^2} - \frac{2j(d-j)}{d^2}\right] \\
= \; & \pi_0 \frac{d^2}{(j+1)^2} \left(\frac{d!}{(d-j)!j!}\right)^2 \left(\frac{d^2 - j^2 - 2jd + 2j^2}{d^2}\right) \\
= \; & \pi_0 \left(\frac{d!}{(d-j)!(j+1)!}\right)^2 (d-j)! \\
= \; & \pi_0 \binom{d}{j+1}^2
\end{aligned}
$$

So $\pi_{d-1} = \pi_0 \binom{d}{d-1}^2 = \pi_0 d^2$

$\pi_d = \pi_0 = \pi_d \binom{d}{d}^2$

Since $\sum \pi_j = 1$ we must have

$$
\pi_0 = \left[\sum_{j=0}^{d} \binom{d}{j}^2\right]^{-1}
$$