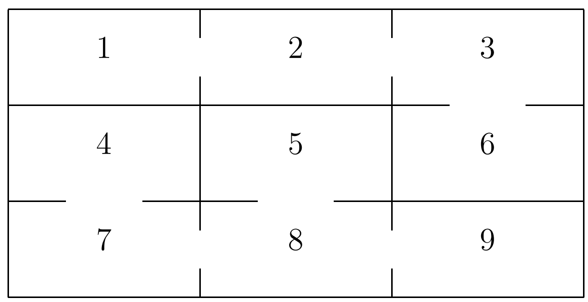
Question

A white rat is put into the maze shown below. The rat moves through the compartments at random i.e. if there are k ways to leave a compartment he chooses each of these with probability $\frac{1}{k}$. Show that the number of the compartment occupied by the rat after n moves is the state for this Markov chain. Write down the transition probability matrix for this Markov chain and identify the closed sets of states. Hence classify the states as null-recurrent or positive recurrent or transient and periodic or aperiodic. What is the effect of introducing a door between compartments 2 and 5?



Answer

The probabilities of moving from one compartment to another depends only on which compartment the rat in occupying. This is the Markov propety. The transition matrix is as follows:

TO / FROM	1	2	3	4		5	6	7	8	9
1	0	1	0	0	⋮	0	0	0	0	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	⋮	0	0	0	0	0
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$	⋮	0	0	0	0	0
4	0	0	1	0	⋮	0	0	0	0	0
	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯
5	0	0	0	0	⋮	0	0	0	1	0
6	0	0	0	0	⋮	0	0	1	0	0
7	0	0	0	0	⋮	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
8	0	0	0	0	⋮	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
9	0	0	0	0	⋮	0	0	0	1	0

There are two closed sets of states {1,2,3,4} and {5,6,7,8,9}. All states are periodic with period 2. Both closed sets form irreducible Markov chains. Hence by a theorem all states are positive recurrent.
If we introduce a door between 2 and 5, all states intercommunicate, so the whole Markov chain is infinite and irreducible, and hence all states are positive recurrent.
The new matrix is

TO / *FROM*	1	2	3	4		5	6	7	8	9
1	0	1	0	0	⋮	0	0	0	0	0
2	$\frac{1}{3}$	0	$\frac{1}{3}$	0	⋮	$\frac{1}{3}$	0	0	0	0
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$	⋮	0	0	0	0	0
4	0	0	1	0	⋮	0	0	0	0	0
	···	···	···	···	···	···	···	···	···	···
5	0	$\frac{1}{2}$	0	0	⋮	0	0	0	$\frac{1}{2}$	0
6	0	0	0	0	⋮	0	0	1	0	0
7	0	0	0	0	⋮	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
8	0	0	0	0	⋮	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
9	0	0	0	0	⋮	0	0	0	1	0

The results could be obtained by "bare hands "methods but would be complicated. Mean recurrence times are also very complicated, especially for states 5 to 9.