

QUESTION

- (i) Express $\tan x$ as a quotient of trigonometric functions and hence, or otherwise, derive the derivative of $\tan x$ with respect to x .
- (ii) State the domain and range of the function $x = \tan^{-1} y$ and sketch its graph.
- (iii) Use the identity $1 + \tan^2 x = \sec^2 x$ to show that

$$\frac{d}{dy}(\tan^{-1} y) = \frac{1}{1 + y^2}.$$

- (iv) Obtain the first and second derivatives with respect to x of the function $u = x \tan^{-1}(x^3)$ and hence verify that u satisfies the equation

$$x^2 \frac{d^2 u}{dx^2} - 4x \frac{du}{dx} + 4u = -\frac{18x^{10}}{(1 + x^6)^2}.$$

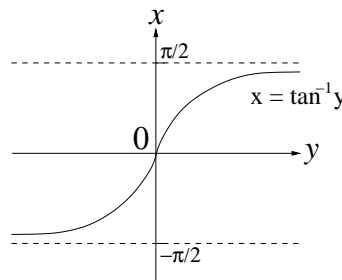
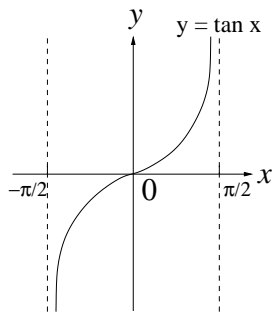
ANSWER

(i) $\tan x = \frac{\sin x}{\cos x}$

Therefore

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

- (ii) If $x = \tan^{-1} y$, then $y = \tan x$.



Domain $-\infty < y < \infty$

Range $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iii) $x = \tan^{-1} y$, $y = \tan x$

$$\text{Therefore } \frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

$$\text{Therefore } \frac{dx}{dy} = \frac{1}{1 + y^2}, \text{ or } \frac{d}{dy}(\tan^{-1} y) = \frac{1}{1 + y^2}$$

(iv) $u = x \tan^{-1}(x^3)$

$$\frac{du}{dx} = \tan^{-1}(x^3) + x \left\{ \frac{1}{1 + (x^3)^2} \cdot 3x^2 \right\} = \tan^{-1}(x^3) + \frac{3x^3}{1 + x^6}$$

$$\begin{aligned} \frac{d^2u}{dx^2} &= \frac{1}{1 + (x^3)^2} \cdot 3x^2 + \frac{(1 + x^6)9x^2 - 3x^3(6x^5)}{(1 + x^6)^2} \\ &= \frac{3x^2}{1 + x^6} + \frac{9x^2(1 - x^6)}{(1 + x^6)^2} \\ &= \frac{3x^2 \{1 + x^6 + 3 - 3x^6\}}{(1 + x^6)^2} \\ &= \frac{3x^2(4 - 2x^6)}{(1 + x^6)^2} \end{aligned}$$

$$\text{i.e. } \frac{d^2u}{dx^2} = \frac{6x^2(2 - x^6)}{(1 + x^6)^2}$$

Hence,

$$\begin{aligned} &x^2 \frac{d^2u}{dx^2} - 4x \frac{du}{dx} + 4u \\ &= \frac{6x^4(2 - x^6)}{(1 + x^6)^2} - 4x \left\{ \tan^{-1} x^3 + \frac{3x^3}{1 + x^6} \right\} + 4x \tan^{-1}(x^3) \\ &= \frac{6x^4(2 - x^6) - 12x^4(1 + x^6)}{(1 + x^6)^2} \\ &= \frac{6x^4(2 - x^6 - 2 - 2x^6)}{(1 + x^6)^2} \\ &= \frac{6x^4(-3x^6)}{(1 + x^6)^2} \\ &= -\frac{18x^{10}}{(1 + x^6)^2} \end{aligned}$$