NOTE REFERENCE TO QUESTION 4

Question

Verify the following integral.

$$\int_0^\infty \frac{\log x}{(x^2+1)^2} = -\frac{\pi}{4}$$

Answer

$J = \oint_C \frac{dz\ \log z}{(z^2+1)^2}$ where C is as in Q4.

This has double poles at $z = \pm i$.

$$\begin{aligned} J &= 2\pi i \times (\text{residue at} + \text{i}) \\ &= 2\pi i \lim_{z\to i}\left[\frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-i)^2\log z}{(z+i)^2(z-i)^2}\right\}\right] = \cdots = \\ &= \left(\frac{\pi}{8}+\frac{i}{4}\right)2\pi i \\ J &= \int_{-R}^{-\epsilon} + \int_{\Gamma_1} + \int_{\epsilon}^{R} + \int_{\Gamma_2} \text{ as above.} \end{aligned}$$

Consider $\lim_{R\to\infty}$ and $\lim_{\epsilon\to 0}$, $\int_{\Gamma_1}$ and $\int_{\Gamma_2} \to 0$

$$J = \int_\infty^0 \frac{d(xe^{i\pi})\log(xe^{i\pi})}{((xe^{i\pi})^2+1)^2} + \int_0^\infty \frac{dx\ \log x}{(x^2+1)^2} = \left(\frac{\pi}{8}+\frac{i}{4}\right)2\pi i$$

$$\Rightarrow 2\int_0^\infty \frac{dx\ \log x}{(x^2+1)^2} + i\pi \underbrace{\int_0^\infty \frac{dx}{(x^2+1)^2}} = \left(\frac{\pi}{8}+\frac{i}{4}\right)2\pi i = -\frac{\pi}{2} + i\frac{\pi^2}{4}$$

simple residue integral with double poles at $x = \pm i \to$ do using $= \frac{\pi}{4}$

PICTURE

$$\Rightarrow 2\int_0^\infty \frac{dx\ \log x}{(x^2+1)^2} + \frac{i\pi^2}{4} = -\frac{\pi}{2} + \frac{i\pi^2}{4}$$

$$\Rightarrow \int_0^\infty \frac{dx\ \log x}{(x^2+1)^2} = -\frac{\pi}{4}$$