Question

Verify the following integral.

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} dz \frac{\exp(zt)}{\sqrt{z}} = \frac{1}{\sqrt{\pi t}},\ \ a > 0$$

Answer

$$I = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} dz \frac{e^{zt}}{\sqrt{z}}$$

PICTURE Now $Re(zt) < 0$ for $t > 0$ $\ Re(z) < 0$

Thus complete contour as:

PICTURE

and consider

$$J = \int_{a-i\infty}^{a+i\infty} + \int_{\Gamma_1} + \int_{Re^{i\pi}}^{\epsilon e i\pi} + \int_{\Gamma_2} + \int_{+\epsilon e^{-i\pi}}^{+Re^{-i\pi}} = 0$$

By Cauchy.

Now consider $\lim_{R\to\infty} J$

$$\lim_{R\to\infty} J = \int_{a-i\infty}^{a+i\infty} +0 + \int_{\infty e^{i\pi}}^{0e^{i\pi}} +0 + \int_{0e^{-i\pi}}^{\infty e^{-i\pi}} = 0$$

By standard tricks.

So

$$
\begin{aligned}
2\pi i I &= -\int_{\infty e^{i\pi}}^{0e^{i\pi}} - \int_{0e^{-i\pi}}^{\infty e^{-i\pi}} \\
&= \underbrace{-\int_{\infty}^{0} \frac{e^{i\pi} e^{xe^{i\pi}t}}{\sqrt{xe^{i\pi}}}\, dx}_{z = xe^{i\pi}} - \underbrace{\int_{0}^{\infty} \frac{e^{-i\pi} e^{xe^{-i\pi}t}}{\sqrt{xe^{-i\pi}}}\, dx}_{z = xe^{-i\pi}} \\
&= \int_0^\infty \frac{e^{-xt}}{\sqrt{x}} dx [e^{i\frac{\pi}{2}} - e^{-\frac{\pi}{2}}] \\
I &= \int_0^\infty \frac{e^{-xt}}{\sqrt{x}} dx \frac{2i}{2\pi i} \\
&= \frac{2}{\pi} \int_0^\infty du\ e^{-tu^2} \\
&= \frac{2}{\pi} \frac{\sqrt{\pi}}{2\sqrt{t}} \\
&= \frac{1}{\sqrt{\pi t}}
\end{aligned}
$$