Question

Verify the following integral.

$$\int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} = \frac{2\pi}{\sqrt{3}}$$

(Hint: There is a branch cut between $x = 0,\ x = 1$.)

Answer

Consider $J = \displaystyle\int_C \frac{dz}{(z^2 - z^3)^{\frac{1}{3}}}$

PICTURE

This is equal to $\displaystyle\int \frac{dz}{(z^2 - z^3)^{\frac{1}{3}}}$ by Cauchy's theorem.

PICTURE

$$\begin{aligned}
\int_{C_1} &= \int_1^0 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} + \underbrace{\int_0^{e^{-2\pi i}} \frac{dz}{(z^2 - z^3)^{\frac{1}{3}}}}_{z = xe^{-2\pi i}\text{ (defining z on } -2\pi < \arg(\text{z}) \leq 0\ (\star))} \\
&= \int_1^0 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} + \int_0^1 \frac{dx}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}} e^{\frac{-2\pi i}{3}}} \\
&= (e^{\frac{2\pi i}{3}} - 1) \int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} \\
&= e^{i\frac{\pi}{3}}(e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}) \int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} \\
&= 2ie^{i\frac{\pi}{3}} \sin\frac{\pi}{3} \times \int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}}
\end{aligned}$$

$$= \quad i\sqrt{3}e^{i\frac{\pi}{3}} \int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}}$$

Now $\int_C dz = \int_0^{2\pi} i \frac{Re^{i\theta}\ d\theta}{(R^2e^{2i\theta} - R^3e^{3i\theta})^{\frac{1}{3}}}$

Now let $R \to \infty$ we still have $\lim_{R\to\infty} J = \int_{C_1} \frac{dz}{(z^2 - z^3)^{\frac{1}{3}}}$

$$\begin{aligned}\lim_{R\to\infty} J &= \lim_{R\to\infty} \frac{iR}{R} \int_0^{2\pi} \frac{d\theta\ e^{i\theta}}{e^{i\theta}} \left(\frac{e^{-i\theta}}{R} - 1\right)^{\frac{1}{3}} \\ &= i \int_0^{2\pi} d\theta\ \underbrace{e^{+i\frac{\pi}{3}}} = 2\pi i e^{+i\frac{\pi}{3}}\end{aligned}$$

$\frac{1}{(-1)^{\frac{1}{3}}}$ with $(-1) = e^{i\pi}$. Why? This is consistent with branch

($\star$) chosen

Thus

$i\sqrt{3}e^{\frac{\pi i}{3}} \int_0^2 1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} = 2\pi i e^{+\frac{i\pi}{3}}$

$\Rightarrow \int_0^1 \frac{dx}{(x^2 - x^3)^{\frac{1}{3}}} = \frac{2\pi}{\sqrt{3}}$