Question
Verify the following integral.
$\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$
Answer
Consider $\oint_C \frac{dz}{z^6+1}$
PICTURE

Now $z^6+1=0$ when
$z = e^{\frac{i\pi}{6}},\ e^{\frac{3i\pi}{6}},\ e^{\frac{5i\pi}{6}},\ e^{\frac{7i\pi}{6}},\ e^{\frac{9i\pi}{6}},\ e^{\frac{11i\pi}{6}}$
being simple poles of the integrand.
Only $e^{\frac{i\pi}{6}},\ e^{\frac{i\pi}{2}},\ e^{\frac{5i\pi}{6}}$ are included in C.
Then we have

$$J = 2\pi i \sum \text{ residues at } \mathrm{e}^{\frac{\mathrm{i}\pi}{6}},\ \mathrm{e}^{\frac{\mathrm{i}\pi}{2}},\ \mathrm{e}^{\frac{5\mathrm{i}\pi}{6}}$$

- $$\begin{aligned} \text{Residue}(\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}) &= \lim_{z\to e^{\frac{i\pi}{6}}} \left[(z - e^{\frac{i\pi}{6}} \frac{1}{z^6+1}\right] \\ &\quad \text{Use l'Hopital!} \\ &= \lim_{z\to e^{\frac{i\pi}{6}}} \left(\frac{1}{6z^5}\right) \\ &= \frac{1}{6} e^{\frac{-5i\pi}{6}} \end{aligned}$$
- $$\begin{aligned} \text{Residue}(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}) &= \lim_{z\to e^{\frac{i\pi}{2}}} \left[(z - e^{\frac{i\pi}{2}} \frac{1}{z^6+1}\right] \\ &\quad \text{Use l'Hopital!} \\ &= \lim_{z\to e^{\frac{i\pi}{2}}} \left(\frac{1}{6z^5}\right) \\ &= \frac{1}{6} e^{\frac{-5i\pi}{2}} \end{aligned}$$

- $$\begin{aligned}
&\text{Residue}(e^{\frac{5i\pi}{6}}) \\
=\ & \lim_{z\to e^{\frac{5i\pi}{6}}} \left[(z - e^{\frac{5i\pi}{6}} \frac{1}{z^6+1}\right] \\
&\text{Use l'Hopital!} \\
=\ & \lim_{z\to e^{\frac{5i\pi}{6}}} \left(\frac{1}{6z^5}\right) \\
=\ & \frac{1}{6} e^{\frac{-25i\pi}{6}}
\end{aligned}$$

Thus

$$\oint \frac{dz}{z^6+1} = 2\pi i \left[\frac{1}{6} e^{\frac{-5i\pi}{6}} + \frac{1}{6} e^{\frac{-5i\pi}{2}} + \frac{1}{6} e^{\frac{-25i\pi}{6}}\right] = \frac{2\pi}{3}$$

Now let $R \to \infty$ and contribution from semicircle, k is given by

$$\begin{aligned}
|k| &= iR \int_0^{\pi} \frac{d\theta}{R^6 e^{i6\theta} + 1} \\
&\leq R \int_0^{\pi} \frac{d\theta}{R^6 e^{6i\theta+1}} \\
&\leq \frac{R}{|R^6 - 1|} \int_0^{\pi} d\theta \\
&= \frac{\pi R}{R^6 - 1}
\end{aligned}$$

Thus $\lim_{R\to\infty} |k| \leq \lim_{R\to\infty} \frac{\pi R}{R^6 - 1} = 0$ so no contribution from semicircle. Thus

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dz}{z^6+1} = \frac{2\pi}{3} \quad &\Rightarrow \quad 2\int_0^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \\
&\Rightarrow \quad \int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}
\end{aligned}$$