Exam Question
Topic: LaplaceODE
Find the solution of the differential equation

$$y'' - y' - 2y = f(x),$$

where

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2; \\ 0 & \text{otherwise,} \end{cases}$$

and where $y(0) = y'(0) = 0$.

Solution
Using the Heaviside function we can write $f(x) = 1 - H(x-2)$.
Transforming the differential equation gives

$$\begin{aligned} y'' - y' - 2y &= 1 - H(x-2) \\ p^2\bar{y} = p\bar{y} - 2\bar{y} &= \frac{1}{p}\left(1 - \mathrm{e}^{-2p}\right) \end{aligned}$$

Rearranging this equation gives

$$\begin{aligned} \bar{y} &= \frac{1}{p(p-2)(p+1)}\left(1 - \mathrm{e}^{-2p}\right) \\ &= \left(-\frac{1}{2p} + \frac{1}{6(p-2)} + \frac{1}{3(p+1)}\right)\left(1 - \mathrm{e}^{-2p}\right) \end{aligned}$$

$$\text{So } \; y = -\frac{1}{2} + \frac{\mathrm{e}^{2x}}{6} + \frac{\mathrm{e}^{-x}}{3} + \left(-\frac{1}{2} + \frac{\mathrm{e}^{2(x-2)}}{6} + \frac{\mathrm{e}^{-(x-2)}}{3}\right) H(x-2).$$