

Exam Question

Topic: LaplaceODE

Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x),$$

where

$$f(x) = \begin{cases} 0 & \text{if } x < 3; \\ 1 & \text{if } x \geq 3, \end{cases}$$

and where $\frac{dy}{dx} = 0$ when $x = 0$ and $y(0) = 1$.

Solution

Using the Heaviside function we can write $f(x) = H(x - 3)$.

Transforming the differential equation gives

$$\begin{aligned} p^2\bar{y} - p + p\bar{y} - 1 &= \frac{e^{-3p}}{p} \\ p(p+1)\bar{y} &= \frac{e^{-3p}}{p} + (p+1) \\ \bar{y} &= \frac{e^{-3p}}{p^2(p+1)} + \frac{1}{p} \\ &= e^{-3p} \left[\frac{1}{p+1} - \frac{1}{p} + \frac{1}{p^2} \right] + \frac{1}{p} \\ \text{So } y(x) &= \left[e^{-(x-3)} - 1 + (x-3) \right] H(x-3) + 1 \end{aligned}$$