QUESTION

(a) State Burnside's lemma, explaining carefully any notation that you use.

(b) Let G be a finite group of order divisible by 3.

(i) Let $X = \{(g,h,k)|g,h,k \in G, ghk = e\}$. Find the number of elements of X.

(ii) The cyclic group $\langle t \rangle$ of order 3 acts on the set $G \times G \times G$ via the rule $t(g,h,k) = (h,k,g)$. Show that this defines an action of $\langle t \rangle$ on X.

(iii) Show that the fixed points of t are precisely the elements of order 3 in G.

(iv) Apply Burnside's lemma to the action of $\langle t \rangle$ on X to show that 3 must divide the number of fixed points for t, and deduce that G must have at least one element of order 3.

ANSWER

(a)

$$r|G| = \sum_{g \in G} |X_g|$$

where G a group acts on a set X, r=number of orbits, $|G|$=number of element in G and for each $g \in G,\ X_g = \{x \in X|gx = x\}$

(b) **(i)** For any $g,h \in G$ there is a unique $k \in G$ with $ghk = e$, so there are $|G|^2$ elements in X.

(ii) It suffices to show that for any $(g,h,k) \in X$ $(h,k,g) \in X$ too, i.e. that $ghk = e \Leftarrow hkg = e$. But $hkg = g^{-1}(ghk)g = g^{-1}eg = e$.

(iii) $t(g,h,k) = (g,h,k) \Leftrightarrow (h,k,g) = (g,h,k) \Leftrightarrow h = g = k$, so the fixed points for t in X are precisely the triples (g,g,g) such that $ggg = e$; i.e. $X_t = \{g \in G|g^3 = e\}$

(iv) $|X_t| = |X_{t^{-1}}| = |X_{t^2}|$ = number of elements of order 3 plus 1 (the identity).

So $r|\langle G \rangle| = |X_e| + |X_t| + |X_{t^2}| = |X| + 2|X_t| = |G|^2 + 2$ (number of elements of order 3 +1)

So 2(number of elements of order 3 +1) is divisible by $|\langle t \rangle| = 3$. Hence number of elements of order 3$\neq 0$.