QUESTION

(a) Show that the set $\{x \in \mathbf{R} | x \neq -1\}$ is a group under the operation * defined by $a * b = a + b + ab$.

(b) The following table in the (incomplete) Cayley table of a group G of order 8.

$*$	p	q	r	s	t	u	v	w
p	t	w						
q	u	t						
r			t					
s				t				
t					t			
u								t
v							t	
w						t		

(i) State the classification of groups of order 8.

(ii) decide, giving your reasons, which of the groups in your classification is isomorphic to the one defined by the Cayley table above.

(c) Write down all the possible cycle structures for elements of S_7, and use this to find all the possible orders for elements of S_7, giving one example of an element for each possible order. Explain the relationship between cycle structures and conjugacy classes there are in S_7.

ANSWER

(a) * is a binary operation since $a * b \in \mathbf{R}$ and $a * b = -1 \Leftrightarrow a + b + ab = -1 \Leftrightarrow -(a+1)b(a+1) \Leftrightarrow a = -1$ or $b = -1$.

* is associative since

$$\begin{aligned}(a * b) * c &= (a + b + ab) + c + ac + bc + abc \\ &= a(b + c + bc) + ab + ac_{+}abc \\ &= a * (b * c)\end{aligned}$$

The identity is 0 and the inverse of a, defined by $\frac{-a}{1+a}$ is an element of the set.

(b) (i) There are 3 abelian groups of order 8, $C_2 \times C_2 \times C_2; C_2 \times C_2$ and C_8. There are 2 non-abelian groups of order 8, D_8 and the Quaternians.

(ii) Since $pq \neq qp$ the group is non-abelian. Since $t^2 = t$, t is the identity element and the group has 5 elements of order 2. It is therefore D_8.

(c)

cycle structure	order	example
$[7]$	7	(1234567)
$[6]$	6	(123456)
$[5, 2]$	10	(12345)(67)
$[5]$	5	(12345)
$[4, 3]$	12	(1234)(567)
$[4, 2]$	4	(1234)(56)
$[4]$	4	(1234)
$[3, 3]$	3	(123)(456)
$[3, 2, 2]$	6	(123)(45)(67)
$[3, 2]$	6	(123)(45)
$[3]$	3	(123)
$[2, 2, 2]$	2	(12)(34)(56)
$[2, 2]$	2	(12)(34)
$[2]$	2	(12)
$[1]$	1	(1)(2)(3)(4)(5)(6)(7)

There is exactly one cycle structure for each conjugacy class so S_7 has 15 conjugacy classes. For example the two elements $(1234)(567)$ and $(abcd)(efg)$ conjugate via the element $(1a)(2b)(3c)(4d)(5e)(6f)(7g)$.