QUESTION

Use the Cauchy Residue Theorem to integrate each of the functions around the circle $|z| = 3$ in the counterclockwise sense. (a) $\frac{e^{-z}}{z^2}$, (b) $\frac{e^{-z}}{(z-1)^2}$, (c) $z^2 e^{\frac{1}{z}}$.

ANSWER

The integral is $2\pi i$ (sum of residues inside $|z| = 3$). In each case there is only one pole inside $|z| = 3$. (a) Pole at $z = 0$; $\frac{e^{-z}}{z^2} = \frac{1}{z^2}(1 - z + \frac{z^2}{2!} + \cdots$. Thus residue at pole is -1 and so integral is $-2\pi i$. (b) Pole at $z = 1$. Put $z-1 = w$. Then $\frac{e^{-z}}{(z-1)^2} = \frac{e^{-w+1}}{w^2} = \frac{e^{-w}}{w^2}e^{-1}$ and so by part (a) the residue at $z = 1$ is $-e^{-1}$ and so the integral is $-2\pi i e^{-1}$. (c) $z^2 e^{\frac{1}{z}} = z^2(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots)$ and so residue at $z = 0$ is $1/6$. Thus integral is $2\pi i/6 = \pi i/3$.