QUESTION

(a) Let a denote a real number, where $-1 < a < 1$. Derive the Laurent expansion

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n}$$

that is valid for $|a| < |z| < \infty$.

(b) By writing $z = e^{i\theta}$ and equating real and imaginary parts, use the result in part (a) to derive the formulae

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2},$$

and

$$\sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}.$$

ANSWER

(a) $\frac{a}{z-a} = \frac{a}{z}(1-\frac{a}{z})^{-1}$. As $|a| < |z|$, we can write this as $\frac{a}{z}(1+\frac{a}{z}+\frac{a}{z}^2+\cdots) = \sum_{n=1}^{\infty} \frac{a^n}{z^n}$.

(b) Just put $e^{i\theta} = \cos\theta + ii\sin\theta$, use De Moivre's Theorem and equate real and imaginary parts.