Question

The Bessel functions $J_n(x)$ are the solutions of

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

are well-behaved at the origin. Transform the equation by changing the variables to $w = y\sqrt{x},\ t = \dfrac{x}{\sqrt{n^2 - \frac{1}{4}}}$ and hence show that the WKB solutions for large n are

$$y \sim \frac{A_\pm}{\sqrt{x}}\left(\frac{x^2}{x^2 - n^2}\right)^{\frac{1}{4}} \exp\left(\pm i\left[(x^2 - n^2)^{\frac{1}{2}} - n\arccos\left(\frac{n}{x}\right)\right]\right\},\ t > 1$$

$$y \sim \frac{B_\pm}{\sqrt{x}}\left(\frac{x^2}{x^2 - n^2}\right)^{\frac{1}{4}} \exp\left(\pm\left[(x^2 - n^2)^{\frac{1}{2}} - n\mathrm{arccosh}\left(\frac{\mathrm{n}}{\mathrm{x}}\right)\right]\right\},\ t < 1$$

Answer

$x^2y'' + xy' + (x^2 - n^2)y = 0$

If $t = \dfrac{x}{\sqrt{n^2 - \frac{1}{4}}},\ \partial_x = \dfrac{1}{\sqrt{n^2 - \frac{1}{4}}}\partial_t;\ \dfrac{w(t)}{t^{\frac{1}{2}}(n^2 - \frac{1}{4})^{\frac{1}{4}}} = y$

so

$$
\begin{aligned}
& x^2\left(\frac{1}{\sqrt{n^2-\frac{1}{4}}}\partial_t\right)\left(\frac{1}{\sqrt{n^2-\frac{1}{4}}}\right)\partial_t\left(\frac{w}{t^{\frac{1}{2}}(n^2-\frac{1}{4})^{\frac{1}{4}}}\right) \\
& +x\left(\frac{1}{\sqrt{n^2-\frac{1}{4}}}\right)\partial_t\left(\frac{w}{t^{\frac{1}{2}}(n^2-\frac{1}{4})^{\frac{1}{4}}}\right) \\
& +\left[t^2\left(n^2-\frac{1}{4}\right)-n^2\right]\left(\frac{w}{t^{\frac{1}{2}}(n^2-\frac{1}{4})^{\frac{1}{4}}}\right)=0 \\
\Rightarrow \quad & t^2\partial_t^2\left(\frac{w}{t^{\frac{1}{2}}}\right)+t\partial_t\left(\frac{w}{t^{\frac{1}{2}}}\right)+\left[t^2\left(n^2-\frac{1}{4}\right)-n^2\right]\left(\frac{w}{t^{\frac{1}{2}}}\right)=0 \\
\Rightarrow \quad & t^2\partial_t\left(\frac{t^{\frac{1}{2}}w'-\frac{1}{2}wt^{-\frac{1}{2}}}{t}\right)+\left(t^{\frac{1}{2}}w'-\frac{1}{2}wt^{-\frac{1}{2}}\right) \\
& +\left[t^2\left(n^2-\frac{1}{4}\right)-n^2\right]\left(\frac{w}{t^{\frac{1}{2}}}\right)=0 \\
\Rightarrow \quad & t^2\frac{[t(\frac{1}{2}t^{-\frac{1}{2}}w'+t^{\frac{1}{2}}w''-\frac{1}{2}w't^{-\frac{1}{2}}+\frac{1}{4}wt^{-\frac{3}{2}})-t^{\frac{1}{2}}w'+\frac{1}{2}wt^{-\frac{1}{2}}]}{t^2} \\
& +\left(t^{\frac{1}{2}}w'-\frac{1}{2}wt^{-\frac{1}{2}}\right)+\left(t^2\left(n^2-\frac{1}{4}\right)-n^2\right)\left(\frac{w}{t^{\frac{1}{2}}}\right)=0 \\
\Rightarrow \quad & t^{\frac{3}{2}}w''+\frac{1}{4}wt^{-\frac{1}{2}}-t^{\frac{1}{2}}w'+\frac{1}{2}wt^{-\frac{1}{2}}+t^{\frac{1}{2}}w'-\frac{1}{2}wt^{-\frac{1}{2}} \\
& +t^{\frac{3}{2}}\left(n^2-\frac{1}{4}\right)w-n^2t^{-\frac{1}{2}}w=0 \\
\Rightarrow \quad & \underline{w''+\left(n^2-\frac{1}{4}\right)\left(1-\frac{1}{t^2}\right)w=0}
\end{aligned}
$$

$(t \neq 0)$

Phew!!

Now we have a WKB-type form of the equation. Obviously things go wrong when $|t|=1$ (WKB-type potential $=0 \Rightarrow$ turning point) so separate into $|t|>1$ and $0<|t|<1$.

$$
w \sim e^{g_0^{(n)}\psi_0(t)+g_1^{(n)}\psi_1(t)+\cdots}
$$

$\{g_r\}$ asymptotic sequence as $nto\infty$

$\psi_r(t)=O(1)$ for $t=O(1)$

Therefore

$(g_0\psi_0''+g_1\psi_1''+\cdots)+(g_0^2\psi_0^{2\prime}+g_1^2\psi_1^{2\prime}+\cdots)+(2g)+(n^2-df14)\left(1-\frac{1}{t^2}\right)=0$

Balance at $O(n^2)$

$\Rightarrow g_0^2\psi_0^{2\prime}+n^2\left(1-\frac{1}{t^2}\right)\sim 0$

$\Rightarrow g_0=n,\ \psi_0^{2\prime}=\frac{1}{t^2}-1 \Rightarrow \psi_0'=\pm\sqrt{\frac{1-t^2}{t^2}}$ depends on whether $|t|<1$ or not.

<u>Next balance</u>:

$$\pm n\left[\left(\frac{1-t^2}{t^2}\right)^{\frac{1}{2}}\right]' + n^2\left(\frac{1}{t^2}-1\right) \pm 2ng_1\sqrt{\frac{1-t^2}{t^2}}\psi_1'$$
$$+\left(n^2-\frac{1}{4}\right)\left(1-\frac{1}{t^2}\right)=0$$
$$\Rightarrow \quad \pm\frac{1\cdot n}{2}\left(\frac{1}{t^2}-1\right)^{-\frac{1}{2}}\left(-\frac{2}{t^3}\right) \pm 2ng_1\sqrt{\frac{1}{t^2}-1}\psi_1' - \frac{1}{4}\left(1-\frac{1}{t^2}\right)=0$$

Must have balance at $O(n)$

$$\Rightarrow \frac{n}{t^3}\left(\frac{1}{t^2}-1\right)^{-\frac{1}{2}} = 2ng_1\sqrt{\frac{1}{t^2}-1}\psi_1'$$

$\Rightarrow g_1 = 1(n^0)$

$$\begin{aligned}
\psi_1' &= \frac{1}{t^3}\left(\frac{1}{t^2}-1\right)^{-1}\\
&= \frac{1}{t^3}\left(\frac{1-t^2}{t^2}\right)^{-1} = \frac{1}{2t(1-t^2)}\\
\text{and} \quad \psi_1 &= \frac{1}{2}\int^t \frac{ds}{s(1-s^2)}\\
&= \frac{1}{2}\int^t ds, \ \left(\frac{1}{s}-\frac{1}{2}\frac{1}{(1+s)}+\frac{1}{2}\frac{1}{(1-s)}\right)\\
&= \frac{1}{2}\ln t - \frac{1}{4}\ln(1+t) - \frac{1}{4}\ln(1-t)
\end{aligned}$$

$$\Rightarrow \psi_1 = \ln\left[\left(\frac{t^2}{1-t^2}\right)^{\frac{1}{4}}\right]$$

Hence $w \sim A_\pm\left(\frac{t^2}{1-t^2}\right)^{\frac{1}{4}}\exp\left\{\pm in\int^t\sqrt{\frac{s^2-1}{s^2}}\,ds\right\}$ $(A_\pm const)$

The i has been pulled out of the $\sqrt{}$ to help what follows. We need to evaluate the integral:

$$\begin{aligned}
\int\sqrt{\frac{s^2-1}{s^2}}\,ds &= -\int\frac{\sin u}{\cos^2 u}\sqrt{\frac{\frac{1}{\cos^2 u}-1}{\frac{1}{\cos^2 u}}}\,du \quad s=\frac{1}{\cos u}\\
&= -\int\frac{\sin^2 u}{\cos^2 u}\,du\\
&= -\int\left(\frac{1}{\cos^2 u}-1\right)du\\
&= -\int\sec^2 u\,du+u\\
&= u-\tan u\\
&= \arccos\left(\frac{1}{s}\right)-s\sqrt{1-\frac{1}{s^2}}\\
&= \arccos\left(\frac{1}{s}\right)-\sqrt{s^2-1}
\end{aligned}$$

Hence $w \sim A_\pm \left(\frac{t^2}{1-t^2}\right)^{\frac{1}{4}} \exp\left\{\pm in\left[\arccos\left(\frac{1}{t}\right) - \sqrt{t^2-1}\right]\right\}$

replace original variable $x = t\sqrt{n^2 - \frac{1}{4}},\ y = \frac{w}{\sqrt{x}}$

$$\begin{aligned} y &\sim \frac{A_\pm}{\sqrt{x}} \left(\frac{x^2}{(n^2-\frac{1}{4})(1-\frac{x^2}{n^2-\frac{1}{4}})}\right)^{\frac{1}{4}} \\ &\quad \exp\left\{\pm in\left[\arccos\left(\frac{x}{\sqrt{n^2-\frac{1}{4}}}\right) - \sqrt{\frac{x^2}{n^2-\frac{1}{4}} - 1}\right]\right\} \\ &\sim \frac{A_\pm}{\sqrt{x}} \left(\frac{x^2}{n^2-\frac{1}{4}-x^2}\right)^{\frac{1}{4}} \\ &\quad \exp\left\{\pm in\left[\arccos\left(\frac{x}{\sqrt{n^2-\frac{1}{4}}}\right) - \frac{\sqrt{x^2-n^2+\frac{1}{4}}}{\sqrt{n^2-\frac{1}{4}}}\right]\right\} \end{aligned}$$

so as $n \to +\infty,\ n^2 - \frac{1}{4} \sim n^2$

Therefore

$y \sim \frac{A_\pm}{\sqrt{x}} \left(\frac{x^2}{n^2-x^2}\right)^{\frac{1}{4}} \exp\left\{\pm i\left[n \arccos\left(\frac{x}{n}\right) - \sqrt{x^2-n^2}\right]\right\} \quad n \to +\infty$

as required $(A_\pm e^{-\frac{i\pi}{4}} A_\mp$ of question)

for $|t| < 1$ use $w \sim B_\pm \left(\frac{t^2}{t^2-1}\right)^{\frac{1}{4}} \exp\left\{\pm n \int_1^t \sqrt{\frac{1-s^2}{s^2}}\right\} ds$

with $s = \frac{1}{\cosh u}$

$\Rightarrow y \sim \frac{B_\pm}{\sqrt{x}} \left(\frac{x^2}{n^2-x^2}\right) \exp\left\{\pm\left[\sqrt{n^2-x^2} - n\text{arccosh}\left(\frac{\text{n}}{\text{x}}\right)\right]\right\},$
$n \to +\infty$