Sturm - Liouville

Sturm - Liouville systems

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + (\lambda g() - l(x))y = 0 \quad a < x < b \tag{1}$$

$$\alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a)\alpha_4 y'(b) = 0 \tag{2}$$

$$\beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b) = 0 \tag{3}$$

The above relations comprise a Sturm-Liouville system: λ is a parameter to be determined. The relation (2) and (3) are linearly independent i.e. the vectors $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$; $(\beta_1, \beta_2, \beta_3, \beta_4)$ are linearly independent.

Examples

1. String with tension $T(x)$ and density $m(x)$ variable along the string, and subject to a transverse restoring force of magnitude $s(x)$ per unit length per unit transverse displacement. For displacement to varying with time as $\cos \omega t$ we find:

   $$\frac{d}{dx}\left(t(x)\frac{dy}{dx}\right) + \left(\frac{\omega^2}{c^2}m(x) - s(x)\right)y = 0$$

   end conditions e.g. $y(0) = 0$, $y(l) = 0$

2. Thermally conducting bar with slowly varying cross section $A(x)$, heat loss along the surface $h(x)$ per unit length, no internal generation of heat. Variable conductivity $K(x)$. Variable heat capacity $c(x)$/unit vol.

   $$\frac{d}{dx}\left[k(x)A(x)\frac{dy}{dx}\right] + \left[pc(x) - h(x)\right]y = 0$$

   for solutions with a time variation αe^{-pt} and appropriate end conditions.

$$y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

$q(x)$, $r(x)$ continuous in $a \le x \le b$
$y(a)$, $y'(a)$ assigned arbitrary.
Define the vector $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$
Let norm $w \equiv ||w|| = |w_1| + |w_2| = o \Leftrightarrow w = 0$
$||w_1|| + ||w_2|| \ge ||w_1 + w_2||$
If $M = \begin{pmatrix} m_1 1 & m_1 2 \\ m_2 1 & m_2 2 \end{pmatrix}$ and $c = 2\max|m_{ij}|$
$||Mw|| \le c||w||$
Define $w'(x) = \begin{pmatrix} w_1'(x) \\ w_2'(x) \end{pmatrix} \int_a^x w(t)dt = \begin{pmatrix} \int_a^x w_1(t)dt \\ \int_a^x w_2(t)dt \end{pmatrix}$

$$\begin{aligned} \left|\left|\int_a^x w(t)\,dt\right|\right| &= \left|\int_a^x w_1(t)dt\right| + \left|\int_a^x w_2(t)dt\right| \\ &\le \left|\int_a^x w_1(t)dt\right| + \left|\int_a^x w_2(t)dt\right| \\ &= \int_a^x ||w(t)dt|| \end{aligned}$$

Now define $v(x) = y'(x)$ then $v'(x) = -(x)v(x) - r(x)y(x)$

define $w(x) = \begin{pmatrix} y(x) \\ v(x) \end{pmatrix}$ (1)

$\dfrac{d}{dx}w(x) = A(x)w(x)$ (2)

where $A(x) = \begin{pmatrix} 0 & 1 \\ -r(x) & q(x) \end{pmatrix}$ (3)

Let $c(x) = 2\max\{1,\ |r(x)|,\ |q(x)|\}\ a \le x \le b$
Let $w(a) = \alpha = \begin{pmatrix} y(a) \\ y'(a) \end{pmatrix}$, $c = \sup\limits_{a \le x \le b} c(x)$
From (2), integrating from a to x

$$w(x) = \alpha + \int_a^x A(t)w(t)dt \tag{4}$$

[This is a vector intrgral equation]
Define the iterant $w^k(x) \quad k = 0, 1, \ldots$ by $w^0(x) = \alpha$
$w^{k+1}(x) = \alpha + \int_a^x A(t) + w^k(t)dt \quad k = 0, 1, \ldots$ (5)

1. By induction the w^k all exist and are continuous in $[a, b]$

2. By induction on the equation

$$w^{k+1}(x) - w^k(x) = \int_a^x A(t)(w^k(t) - w^{k-1}(t))dt$$

we can show that

$$||w^{k+1}(x) - w^k(x)|| \leq ||\alpha|| \frac{[c(x-a)]^{k+1}}{(k+1)!}$$

3. We then have

$\sum_{k=0}^{\infty} ||w^{k+1}(x) - w^k(x)||$ converges uniformly in $[a, b]$

therefore $\sum_{k=0}^{\infty} (w^{k+1}(x) - w^k(x))$ converges uniformly in $[a, b]$

$w^k(x) = \alpha + \sum_{r=0}^{k-1} (w^{r+1}(x) - w^r(x))$

Hence $\lim_{k \to \infty} w^k(x)$ exists $= w(x)$ uniformly in $[a, b]$ and $w(x)$ is continuous in $[a, b]$ letting $k \to \infty$ in equation (5) we get

$$w(x) = \alpha + \int_a^x A(t)w(t)dt$$

RHS is differentiable , therefore

$$w'(x) = A(t)w(x) \quad \text{amd} \quad w(\alpha) = \alpha$$

Uniqueness

Assume that $z(x)$ is a solution, continuous and bounded in $[a, b]$, of the integral equation

$$\begin{aligned} z(x) &= \alpha + \int_a^x A(t)z(t)dt \\ w^{k+1}(x) &= \alpha + \int_a^x A(t)w^k(t)dt \\ w^{k+1}(x) - z(x) &= \int_a^x A(t)(w^k(t) - z(t))dt \end{aligned}$$

By induction we can show that

$$||w^k(x) - z(x)|| \leq \frac{m[c(x-a)]}{k!}$$

where $m = \bar{bd}||z(x) - a||$ in $[a, b]$
Therefore $\lim_{k\to\infty} ||w^k(x) - z(x)|| = 0$ uniformly in $[a, b]$
Therefore $||w(x) - z(x)|| = 0$ i.e. $|w(x) = z(x)$
($p_0 > 0$, $p_0 p_1 p_2$ continuous in $[a, b]$)
A solution of $L(y) = 0$ exists in $[a, b]$ such that $y(v)$, $y'(c)$ have arbitrary values, $c \in [a, b]$ and this solution is unique.

Wronskian of two solutions

If $u(x)$, $v(x)$, $u'(x)$, $v'(x)$ are continuous then

$$W =_{af} \begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}$$

is the Wronskian determinant.

(i) $W = 0$ in $[a, b]$ is the necessary and sufficient condition that u and v are linearly dependent.

(ii) $W \neq 0$ in $[a, b]$ is the necessary and sufficient condition that u and v are linearly independent.

If now $L(u) = 0$ $L(v) = 0$

$$\begin{aligned} 0 &= vL(u) - uL(v) \\ &= p_0(vu'' - v''u) + p_1(vu' - v'u) \\ &= p_0 \frac{d}{dx}(vu' - v'u) + p_1(vu' - v'u) \end{aligned}$$

Write

$$p(x) = \exp \int_a^x \frac{p_1(t)}{p_0(t)} dt$$

therefore $\dfrac{p'(x)}{px} = \dfrac{p_1(x)}{p_0(x)}$
Therefore the above equation is

$$p\frac{d}{dx}(vu' - v'u) + p'(vu' - v'u) = 0$$

i.e. $p(vu' - v'u)$ =constant.
i.e. $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$ =constant.

$\int_a^x \frac{p_1(t)}{p_0(t)}\,dt$ is bounded in $[a.b]$.
Therefore $p(x) = \exp \int_a^x \frac{p_1(t)}{p_0(t)} dt > 0$ in $[a, b]$
Hence

(i) $W = 0$ at $x = c \in [a, b] \Rightarrow W = 0 \;\; a \le x \le b$

(ii) $W \ne 0$ at $x = c \in [a, b] \Rightarrow W \ne 0 \;\; a \le x \le b$

Example of choice of linearly independent solutions.
$u(x): \;\; u(a) = 1 \;\;\; u'(a) = 0$
$v(x): \;\; v(a) = 0 \;\;\; v'(a) = 1$
$W(u, v) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \ne 0$ therefore u, v, are linearly independent in $a \le x \le b$

Fundamental System of solutions

Definition Any pair $u(x)$, $v(x)$ of linearly independent solutions constitute a fundamental system.

Theorem Any solution of $L(y) = 0$ is of the form $y = Au + Bv$

Proof We can choose A, B such that $y(c)$, $y'(c)$ have any assigned values, c in $[a, b]$

$$\begin{aligned} y(c) &= Au(c) + Bv(c) \\ y'(c) &= Au'(c) + Bv'(c) \end{aligned}$$

$\begin{vmatrix} u(c) & v(c) \\ u'(c) & v'(c) \end{vmatrix} \ne 0$ therefore A and B are uniquely determined.
Consider $z(x) = y(x) - Au(x) - Bv(x)$
$L(z) = 0$ -(i) as L is linear.
$\left.\begin{matrix} z(c) = 0 \\ z'(c) = 0 \end{matrix}\right\}$ -(ii)
(i) and (ii) are satisfied by $z \equiv 0$, Therefore by the uniqueness theorem this is the only solution.
Therefore $z \equiv 0 \quad a \le x \le b$
Therefore $y(x) = A \cdot u(x) + B \cdot v(x) \quad a \le x \le b$

Adjoint (2nd order) linear differential operators

$$L(u) = (p_0 D^2 + p_1 D + p_2)u \tag{1}$$

Let $v = v(x)$, $v'(x)\, v''(x)$ exist.

$$\begin{aligned} vL(u) &= vp_0 D^2 u + vp_1 Du + vp_2 u \\ vp_0 D^2(u) &= uD^2(vp_0) + D[vp_0 Du - uD(vp_0)] \\ vp_1 Du &= -uD(vp_1) + D(vp_1 u) \end{aligned}$$

Hence $vL(U) = u[D^2(vp_0) - D(vp_1) + vp_2] + D[vp_0 Du - uD(vp_0) + vp_1 u]$
Write $M(v) = D^2(vp_0) - D(vp_1) + vp_2$ (2)
$M(v) = p_0 D^2 v + (2p_0' - p_1)Dv + (p_0'' - p_1' + p_2)v$ (3)

$$\begin{aligned} vL(u) - uM(v) &= \frac{d}{dx}(vp_0 u' - u(p_0 v' + p_0' v) + p_1 uv) \\ &= \frac{d}{dx}(p_0(vu' - u'v) + (p_1 - p_0')uv) \end{aligned} \tag{4}$$

M is said to be the adjoint of L in view of the form of R.H.S. Also $L = adj M$

Self adjoint Operator

<u>Definition</u> L is self adjoint if $M \equiv L$

From (1) and 93) the necessary and sufficient condition is $p_1 = p_0'$
then $L(u) = \dfrac{d}{dx}\left(p_0 \dfrac{du}{dx}\right) + p_2 u$
If L is self adjoint the from (4) $vL(u) - uL(v) = \dfrac{d}{dx} p_0(vu' - u'v)$

Reduction to self-adjoint form

if $L = p_0 D^2 + p_1 D + p_2$
Define $p(x) = \exp \displaystyle\int_a^x \frac{p_1(t)}{p_0(t)} dt$ ($p_0 > 0$ in $[a, b]$)
Then $\dfrac{p'(x)}{p(x)} = \dfrac{p_1(x)}{p_0(x)}$
Hence

$$\begin{aligned} \frac{p}{p_0} L(y) &= \left(pd^2 + p'D + \frac{pp_2}{p_0}\right) y \\ &= \frac{d}{dx}\left(p \frac{dy}{dx}\right) + \frac{pp_2}{p_0} y \end{aligned}$$

Self adjoint System

$$L(y) = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy$$

Suppose we have a self adjoint operator L. Consider the homogeneous system:

$L(y) = 0 \qquad (a \leq x \leq b)$ (1)

$$\left.\begin{array}{l} 0 = U_1(y) = a_1y(a) + b_1y'(a) + c_1y(b) + d_1y(b) \\ 0 = U_2(y) = a_2y(a) + b_2y'(a) + c_2y(b) + d_2y(b) \end{array}\right\} \quad (2)$$

[condition (2) constitutes 2-point boundary conditions]

Let u, v be any two functions such that $u'v'$ are continuous in $[a, b]$ (not necessarily satisfying $L(y) = 0$)

$vL(u) - uL(v) = \frac{d}{dx}p(vu' - v'u)$

Therefore

$$\int_a^x (vLu - uLv)dx = -p(b)\begin{vmatrix} u(b) & v(b) \\ u'(b) & v'(b) \end{vmatrix} + p(a)\begin{vmatrix} u(a) & v(a) \\ u'(a) & v'(a) \end{vmatrix} \quad (3)$$

Definition The boundary conditions (2) are said to be self-adjoint if R.H.S of (3) vanishes whenever u and v satisfy (2) i.e. $U_i(v) = 0$, $U_i(v) = 0\, i = 1, 2$ and u and v are linearly independent.

Theorem The necessary and sufficient condition for this to occur is

$$p(a)\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = p(b)\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad (4)$$

Proof The equations $U_i = 0$ $V_i = 0$ may be written:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} + \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = 0$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} v(a) \\ v'(a) \end{pmatrix} + \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} v(b) \\ v'(b) \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} u(a) & v(a) \\ u'(a) & v'(a) \end{pmatrix} + \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} u(b) & v(b) \\ u'(b) & v'(b) \end{pmatrix} = 0$$

$AW_a + BW_b = 0$

Therefore $AW_a = -BW_b$

Taking determinants:

$|A||W_a| = |B||W_b|$ (5)

($|-B| = |B|$ as B is of even order)

Therefore $p(a)|W_a| = p(b)|W - b|$

$\Leftrightarrow p(a)|B| = p(B)|A|$

Examples

(i) $\begin{matrix} y(a)=0 \\ y(b)=0 \end{matrix}$ $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(string with fixed ends.)

(ii) $\begin{matrix} y'(a)=0 \\ y'(b)=0 \end{matrix}$ $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(string with free ends.)

(iii) $\begin{matrix} y(a)=0 \\ y'(b)=0 \end{matrix}$ $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(string with 1 fixed end, and 1 free end.)

(iv) $\begin{matrix} y'(a)+\alpha y(a)=0 \\ y'(b)+\beta y(b)=0 \end{matrix}$ $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & 1 \end{pmatrix}$

(elasticity constrained ends.)

(v) $\begin{matrix} y(a)-y(b)=0 \\ y'(a)-y'(b)=0 \end{matrix}$ $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(Periodic boundary conditions.)

In (i) - (iv) $|A| = |B| = 0$
In (v) $|A| = |B| = 1$ and we require $p(a) = p(b)$

Sturm Loiuville Systems

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + (\lambda q(x) - r(x))y = 0 \qquad (1)$$

where λ is a parameter. i.e. $L(y) - \lambda q(y) = 0$ (2)

$$L(y) = \frac{d}{dx}\left(p\frac{dy}{dx}\right) - r(x)y$$

We assume $p > 0$ in $[a, b]$ (later also $r > 0$, $q > 0$)
The boundary conditions are:

$$U_i(y) = a_1 y(a) + b_i y'(a) + c_i y(b) + d_i y(b) = 0, \quad i = 1, 2 \qquad (3)$$

The system is assumed to be self adjoint.

Eigenvalues and Eigenfunctions

Let , v be any linearly independent pair of solutions of
$L(y) + \lambda q \cdot y = 0$ (1)
Then any other solution y is a linear combination of u and v.
$y(x) = \alpha u(x) + \beta v(x)$ where α and β are constants. Now U_1 and U_2 are linear and homogeneous.

$$\begin{aligned} U_1 &= \alpha U_1(U) + \beta U_1(v) \\ U_2 &= \alpha U_2(U) + \beta U_2(v) \end{aligned}$$

$U_1(y) = 0 \quad U_2(y) = 0$ gives

$$\begin{pmatrix} U_1(u) & U_1(v) \\ U_2(u) & U_2(v) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \qquad (2)$$

For a non trivial solution for α and *beta* the determint must vanish.

$$\Delta(\lambda) = \begin{vmatrix} U_1(u) & U_1(v) \\ U_2(u) & U_2(v) \end{vmatrix} = 0 \qquad (3)$$

The determinant is a function of λ since both y and v satisfy $L(y) + \lambda q y = 0$
i.e. $u = u(x, \lambda) \quad v = v(x, \lambda)$
$U_i(u) = a_i u(a, \lambda) + b_i u_x(a, \lambda) + c_i u(b, \lambda) + d_i u_x(b, \lambda)$
and
$U_i(v) = a_i v(a, \lambda) + b_i v_x(a, \lambda) + c_i v(b, \lambda) + d_i v_x(b, \lambda)$
The equation $\Delta(\lambda) = 0$ is the characteristic equation for λ

Definition the value of λ satisfying $\Delta(\lambda) = 0$ are the eigenvalue of the system.

When $\lambda = \lambda_n$ (an eigenvalue) there is a non-trivial solution for α, β from (2) and the solution $y = \phi_n = \alpha_n u(x, \lambda_n) + \beta_n(x, \lambda_n)$ is said to be the eigenfunction belonging to λ_n. ϕ_n is uniquely determined apart from any non-zero constant.
We assume
(1) There exists an infinite set $\lambda_1, \lambda_2, \ldots$ of eigenvalues.
(2) there exist only one linearly independent eigenfunction belonging to λ_n. This may not be true in special cases)
Example

1. Suppose the boundary condition is $y(a) = 0$, then $y'(a) \neq 0$. Suppose ϕ' ϕ^2 were 2 linearly independent eigenfunctions belonging to λ.

   Let $\phi^3 = \phi' - \phi^2 \dfrac{\phi'(a)}{\phi^2(a)}$

   $\phi^3(a) = 0$ as $\phi'(a) = \phi^2(a) = 0$

Also $\phi'^3(a) = 0$. Therefore as $L(\phi^3) + \lambda q(\phi^3) = 0$

$\phi^3(x) = 0$ in the whole interval. Therefore $\phi' = \phi^2$ therefore (2) holds.

2.

$$\begin{aligned} y'' + \lambda y &= 0 \quad p = 1 \; q = 1 \\ y(0) &= y(l) \\ y'(0) &= y'(l) \end{aligned}$$

The solution of the differential equation is

$y = A\cos x\lambda^{\frac{1}{2}} + B\sin x\lambda^{\frac{1}{2}}$

Then

$$\begin{aligned} A &= A\cos l\lambda^{\frac{1}{2}} + B\sin l\lambda^{\frac{1}{2}} \\ \lambda^{\frac{1}{2}}B &= \lambda^{\frac{1}{2}}(-A\sin l\lambda^{\frac{1}{2}} + B\sin l\lambda^{\frac{1}{2}}) \end{aligned}$$

Hence (rejecting $\lambda = 0$ as trivial) we have

$$\begin{aligned} \begin{vmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{vmatrix} &= 0 \quad \theta = l\lambda^{\frac{1}{2}} \\ (1 - cos\theta)^2 + \sin^2\theta &= 0 \\ 2(1 - \cos\theta &= 1 \quad \theta = \pm 2n\pi \;\; n = 1, 2, \ldots \\ \lambda_n &= \frac{4\pi^2}{l^2}n^2 \quad n = 1, 2, \ldots \end{aligned}$$

For $\lambda = \lambda_n$ the equations a for A and B are

$$\begin{aligned} 0A + 0B = 0 \\ 0A + 0B = 0 \end{aligned}$$

i.e. A and B are arbitrary.

Therefore $y = A\cos\frac{2n\pi x}{l} + B\frac{2n\pi x}{l}$ is an eigenfunction belonging to λn

i.e. $\cos\frac{2n\pi x}{l}, \sin\frac{2n\pi x}{l}$ are eigenfunctions belonging to λ_n and are linearly independent: (2) doesn't hold.

Independence of eigenvalues with respects to the choice of u and v

Let $\bar{u}, \bar{v}$ be another linearly independent pair of solutions of $L(y) + \lambda q \cdot y = 0$

Then $\begin{array}{rcl} u &=& c_{11}\bar{u} + c_{12}\bar{v} \\ v &=& c_{21}\bar{u} + \_22\bar{v} \end{array}$ $c_{11}, \ldots$ constant and $\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0$

$$\begin{array}{rcl} U_i(u) &=& c_{11}U_i(\bar{u}) + c_{12}U_i(\bar{u}) \\ U_i(v) &=& c_{21}U_i(\bar{v}) + c_{22}U_i(\bar{v}) \quad i = 1, 2 \end{array}$$

i.e. $\begin{pmatrix} U_1(u) & U_2(u) \\ U_1(v) & U_2(v) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} U_1(\bar{u}) & U_2(\bar{u}) \\ U_1(\bar{v}) & U_2(\bar{v}) \end{pmatrix}$

Taking determinants $\Delta(\lambda) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \begin{vmatrix} U_1(\bar{u}) & U_2(\bar{u}) \\ U_1(\bar{v}) & U_2(\bar{v}) \end{vmatrix} = |C|\bar{\Delta}(\lambda)$

and $|C| \neq 0$ therefore $\Delta(\lambda) = 0 \Leftrightarrow \bar{\Delta}(\lambda) = 0$

Example Uniform string under constant tension and with fixed ends, and no restraining force.

The differential equation is

$$y'' + \lambda y = 0 \qquad \lambda = \frac{w^2}{c^2}$$

$a = 0,\, b = l \Rightarrow \begin{array}{l} U_1(y) = y(0) = 0 \\ U_2(y) = y(l) = 0 \end{array}$

Two linearly independent solutions of the equation are

$$u = \cos x\lambda^{\frac{1}{2}} \qquad v = \sin x\lambda^{\frac{1}{2}}$$

$u(0) = 1 \quad u'(0) = 0 \quad v(0) = 0 \quad v'(0) = \lambda^{\frac{1}{2}}$

$\Delta(\lambda) = \begin{vmatrix} 1 & 0 \\ \cos\lambda^{\frac{1}{2}} & \sin l\lambda^{\frac{1}{2}} \end{vmatrix} = \sin l\lambda^{\frac{1}{2}}$

Therefore the equation for λ is $\sin l\lambda^{\frac{1}{2}} = 0 \Rightarrow \lambda = \frac{n^2\pi^2}{l^2} \cdot \frac{w}{c} = \frac{n\pi}{l}$

Properties of eigenvalues and Eigenfunctions

We assume now that $q(x) > 0$, $r(x) > 0$ in $[a, b]$ in addition to $p(x) > 0$ in $[a, b]$

1. The eigenvalues are real
2. If $\phi_m\, \phi_n$ belong to $\lambda_m\, \lambda_n$

$$\int_a^b q(x)\phi_n(x)\phi_m(x)dx = 0 \quad (m \neq n)$$

   i.e. $\phi_1,\, \phi_2, \ldots$ are orthogonal over $[a, b]$ with weighting function $q(z)$

3. if the boundary conditions are suitably restricted (and $p, q > 0$ $r \geq 0$ in $[a, b]$) then the eigenvalues are positive

Proofs

1.

2. Let λ_n, λ_n be any two eigenvalues and let ϕ_m, ϕ_n belong to them. Then

$$\begin{aligned} L(\phi_m) + \lambda_m q \phi_m &= 0 \\ L(\phi_n) + \lambda_n q \phi_n &= 0 \\ U_i(\phi_m) = 0 \quad U_i(\phi_n) = 0 & \qquad\qquad i = 1, 2 \end{aligned}$$

$$\int_a^b \{\phi_n L(\phi_m) - \phi_m L(\phi_n)\} dx = p(x)[\phi_n \phi_m' - \phi_m \phi_n']_a^b$$

and the R.H.S =0 if the boundary conditions are self adjoint.

Therefore $\int_a^b b\{\phi_n L(\phi_m) - \phi_m L(\phi_n)\} dx = 0$

Therefore $\int_a^b \{\phi_m \lambda_n q \phi_n - \phi_n \lambda_m q \phi_m\} dx = 0$

Therefore $(\lambda_n - \lambda_m) \int_a^b q \phi_m \phi_n \, dx = 0$ (i)

Therefore $\int_a^b q \phi_m \phi_n \, dx = 0 \qquad m \neq n$ (ii)

(i) If $\lambda = \rho + i\sigma$ is an eigenvalue and $\phi = X + iY$ is an eigenfunction belonging to it then $\bar{\lambda}$ is also an eigenvalue and $\bar{\phi}$ will belong to it.

For $L(\phi) + \lambda q \phi = 0$

$\Rightarrow \overline{L(phi)} + \overline{\lambda q \cdot q} = 0$

$\Rightarrow L(\bar{\phi}) + \bar{\lambda} q \cdot \bar{\phi} = 0$

Since L is a real linear operator and q is real.

Also $U_i(\phi) = 0 \Rightarrow U_i(\bar{\phi}) = 0$ as U_i is real and linear.

Hence in (i) above take $\lambda = \lambda_n$ and $\bar{\lambda} = \lambda_m$

Then

$$\begin{aligned} (\bar{\lambda} - \lambda \int_a^b q \phi \bar{\phi} dx &= 0 \\ (\bar{\lambda} - \lambda) \int_a^b q |\phi|^2 dx &= 0 \quad q > 0 \;\; |\phi| \not\equiv 0 \end{aligned}$$

therefore $\bar{\lambda} - \lambda = 0$ i.e. λ is real.

3.

$$L(\phi_n) + \lambda q \cdot \phi_n = 0$$

Therefore $\lambda_n \int_a^b q\phi_n^2 dx = -\int_a^b \phi_n L(\phi_n) dx$

$$\begin{aligned} \phi_n L(\phi_n) &= \phi \left(\frac{d}{dx} p \frac{d\phi_n}{dx} - r\phi_n \right) \\ &= \frac{d}{dx} \phi_n p \frac{d\phi_n}{dx} - p \left(\frac{d\phi_n}{dx} \right)^2 - r\phi_n^2 \end{aligned}$$

therefore $\lambda_n \int_a^b a\phi_n^2 dx = \int_a^b (p\phi_n^{2'} + r\phi_n^2) dx - [p\phi_n\phi_n']_a^b$

Therefore if the boundary conditions are such that

$[p\phi_n\phi_n']_a^b \leq 0 \quad \lambda_n > 0$

Examples

(i) $y(a) = 0 \quad y(b) = 0 \Rightarrow \phi_n(a) = \phi_n(b) = 0$ and $[p\phi_n\phi_n']_a^b = 0$

(ii) $y'(a) = 0 \quad y'(b) = 0 \Rightarrow \phi_n'(a) = \phi_n'(b) = 0$ and $[p\phi_n\phi_n']_a^b = 0$

(iii)

$$\begin{aligned} y'(a) - h_1 y(a) &= 0 \qquad h_1 > 0 \\ y'(b) + h_2 y(b) &= 0 \qquad h_2 > 0 \\ -[p\phi_n\phi_n']_a^b &= -p(b)(-h_2\phi_n^2(b)) + p(a)(h_1\phi_n^2(a)) > 0 \end{aligned}$$

(iv) $y(a) = y(b) \qquad y'(a) = y'(b)$

Here, with the condition for self adjointness $p(a) = p(b), [p\phi_n\phi_n']_a^b = 0$

Formal Explanations in Eigenfunctions

Consider the homogeneous system

$$L(y) + \lambda q \cdot y = 0 \qquad \left(L = \frac{d}{dx} p \frac{d}{dx} + r\right)$$

$$U_i(y) = 0 \qquad i = 1, 2.$$

with eigenvalues $\lambda_1, \lambda_2, \ldots$
and eigenfunctions $\phi_1, \phi_2 \ldots$
We assume that the ϕ_n have been normalised

$$\text{i.e.} \quad \int_a^b q\phi_n^2 \, dx = 1$$

If a function $F(x)$ defined in $[a, b]$ has a uniformly convergent expansion

$$F(x) = \sum_1^\infty A_n \phi_n(x)$$

then

$$\int_a^b q(x)\phi_m(x)F(x)dx = A_m$$

Consider the non-homogeneous system

$$\begin{aligned} L(y) + \lambda q \cdot y &= f(x) \\ U_i(y) &= 0 \qquad i = 1, 2 \end{aligned}$$

1. If λ is not an eigenvalue of the homogeneous system the solution is unique.
2. If $\lambda = \lambda_m$ a necessary condition for existence of a solution is that

   $$\int_a^b \phi_m(x) f(x) dx = 0$$

   i.e. f must be orthogonal to ϕ_m.
3. if $\lambda = \lambda_m$ and f is orthogonal to ϕ_m, then the solution is not unique.

Proofs

1. Suppose y and z are two solutions

$$\begin{array}{rclrcll} L(y)+\lambda q\cdot y &=& f(x) & U_i(y) &=& 0 & i=1,2 \\ L(z)+\lambda q\cdot z &=& f(x) & U_i(z) &=& 0 & i=1,2 \\ L(y-z)+\lambda q\cdot(y-z) &=& 0 & U_i(y-z) &=& 0 & i=1,2 \end{array}$$

   i.e. $L(y-z)$ is a solution of the homogeneous system. If λ is not an eigenvalue this must be zero. Therefore $y=z$ in $[a,b]$

3. If $\lambda=\lambda_m$, $y-z=A\phi_m(x)$ where A is an arbitrary constant i.e. $y=z+A\phi_m(x)$ and the solution is not unique.

2. $\int_a^b [\phi_m L(y)-yL(\phi_m)]dx = [p(x)[\phi_m y'-\phi_m' y]]_a^b = 0$

   since $y_v\phi_m$ satisfies the boundary conditions which are self adjoint. i.e.
   $\int_a^b [\phi_m\{f(x)-\lambda q\cdot y\}-y\{-\lambda_m q\phi_m\}]dx = 0$

   $$\lambda-\lambda_n\int_a^b q\phi_m y dx = \int_a^b \phi_m f\, dx \qquad (1)$$

   Therefore λ is an eigenvalue i.e. $\lambda=\lambda_m$,

$$\int_a^b \phi_m f\, dx = 0$$

Formal Series Solution

If we assume y has an expansion in eigenfunction we have from (1) above.

$$(\lambda-\lambda_n)a_n = \int_a^b \phi_n f\, dx = b_n$$

hence if λ is <u>not</u> an eigenvalue $a_n = \dfrac{b_n}{\lambda-\lambda_n} \quad n=1,2,\ldots$

i.e.$y = \sum_{n=1}^{\infty} \dfrac{\phi_n(x)}{\lambda-\lambda_n}\int_a^b \phi_n(\xi)f(\xi)\,d\xi$

If $\lambda=\lambda_m$ then $a_n = \dfrac{b_n}{\lambda_m-\lambda_m} \quad m\neq n$

and $0\cdot a_m = 0$ Therefore

$$\sum_{\substack{n=1\\ n\neq m}}^{\infty} \frac{\phi_n(x)}{\lambda_m-\lambda_n}\int_a^b \phi_n(\xi)f(\xi)\,d\xi + A\phi_m(x), \quad \text{A arbitrary}$$

Stationary Property of Eigenvalues

When the boundary conditions are such that $[p(x)yx)y'(x)]_a^b = 0$ we have

$$\lambda_n = \frac{\int_a^b (p\phi_n'^2 + r\phi_n^2)dx}{\int_a^b q\phi_n^2 dx}$$

Write

$$\left.\begin{array}{rcl} I(\phi,\psi) & = & \int_a^b (p\phi'\psi' + r\phi\psi)dx \\ \\ J(\phi,\psi) & = & \int_a^b q\phi\psi \end{array}\right\} \qquad (1)$$

$\lambda_n = \dfrac{I(\phi_n,\phi_n)}{J(\phi_n,\phi_n)}$

We suppose that ϕ_n are normalised i.e. $J(\phi_n,\phi_n) = 1$

then $\lambda_n = I(\phi_n.\phi_n)$ (2)

Consider $\lambda_n = I(\phi,\phi)$ where $J(\phi, phi) = 1$ (3)

and

(i) ϕ' continuous in $[a,b]$

(ii) ϕ satisfied the boundary conditions

(it does not necessarily satisfies $L(y) + \lambda q(y) = 0$)

We show that $\lambda = I(\phi,\phi)$ is stationary for small variations of ϕ from ϕ_n. This is the extremum property of the integral $I(\phi,\phi)$ subject to the normalising condition $j(\phi, \phi = 0$ and $to(i),(ii)$.

Write $\phi(x) = \phi_n(x) + \epsilon\psi(x)$, where ϵ is a constant and ψ satisfies (i) and also the boundary conditions since U_i is linear. We show that

$$I(\phi\phi) - I(\phi_n\phi_n) = O(\epsilon^)$$

The normalising condition on ϕ is

$1 = J(\phi, phi) = J(\phi_n\phi_n) + 2\epsilon J(\phi_n\psi) + \epsilon^2 J(\psi\psi)$

$1 = 1 + 2\epsilon J(\phi_n\psi) + \epsilon^2 J(\psi\psi)$

Therefore $2\epsilon J(\phi_n\psi) + \epsilon^2 J(\psi\psi) = 0$

$$\begin{array}{rcl} I(\phi\phi) & = & I(\phi_n\phi_n) 2\epsilon I(\phi_n\psi) + \epsilon^2 I(\psi\psi) \\ I(\phi_n\psi) & = & \int_a^b (p\phi_n'\psi_n' + r\phi_n\psi)dx \\ & = & [p\phi_n'\psi_n']_a^b + \int_a^b \left\{-\psi\frac{d}{dx}p\phi_n' + r\phi_n\psi\right\} \\ 0 = [p\phi\phi']_a^b & = & [p\phi_n\phi_n']_a^b + [\epsilon p(\phi_n\psi' + \phi_n'\psi)]_a^b + [\epsilon^2 p\phi\phi']_a^b \end{array}$$

From the self adjoint condition

$$[p(\phi_n\psi - \psi_n'\phi)]_a^b = 0$$

Therefore

$$[p\phi_n'\psi]_a^b = [p\phi_n\psi']_a^b = 0$$

Therefore

$$\begin{aligned} I(\phi_n\psi) &= \int_a^b \psi[-L(\phi_n)]_a^b = 0 \\ &= \lambda_n \int_a^b q\psi\phi_n \, dx \\ &= \lambda_n J(\phi_n\psi) \end{aligned}$$

Therefore

$$\begin{aligned} I(\phi\phi) - I(\phi_n\phi_n) &= 2\epsilon\lambda_n J(\phi_n\psi) + \epsilon^2 I(\psi\psi) \\ &= \epsilon^2[I(\psi\psi)\lambda_n J(\psi\psi)] \end{aligned}$$

This establishes the stationary property.

Illustration

$$y'' + \lambda y = 0 \quad y(0) = 0 \quad y(1) = 0$$

The exact solution for λ_1 and ϕ_1 is $\phi_1 x \sin\pi x \quad \lambda_1 = \pi^2 \approx 9.87$
The normalised $\phi_1 is 2^{\frac{1}{2}} \sin\pi x$
Take $\phi = Cx(1-x)$
$\int_0^1 \phi^2 \, dx = C^2 \int_0^1 x^2(1-x)^2 = C^2 \frac{\Gamma(3)\Gamma(3)}{\Gamma(6)} = \frac{C^2}{30}$
Therefore $\phi = \sqrt{30}x(1-x)$
$I(\phi\phi) = \int_0^1 \phi'^2 \, dx = 30 \int_0^1 (1-2x)^2 \, dx = 30\frac{1}{6}2 = 10$
Compare with 9.87 thus the error is $\approx 1.4\%$

Formulation of the eigenvalue problem as an "isoperimetric problem"

The eigenvalues of the system $L(y) + \lambda q \cdot y = 0$ with $y(a) = y(b) = 0$ are the extrema of $I = \int_a^b (p\phi'^2 + r\phi^2)dx$

subject to the normalising condition $J = \int_a^b q\phi^2 dx = 1$ and

(i) ϕ' continuous in $[a, b]$

(ii) $\phi(a) = \phi(b) = 0$