Bessel Functions

Vibrations of a Membrane

The governing equation for the displacement $w(x, y, t)$from the equilibrium position (the plane $z = 0$) is

$$\nabla_1^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

[The assumptions are:-

(i) That the action across the element $DeltaS$ is a force $T\Delta S$ perpendicular to ΔS, where $T' \to T$ as $\Delta S \to 0$

(ii) the displacement w of any point of the membrane is purely transverse.

(iii) that $\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{1}{2}}$ is small.]

It can be shown that the tension is isotropic at a point at a point (i.e. independent of the orientation of ΔS), independently of (ii) and (iii), and for equilibrium or motion from (ii) it can be shown that T is uniform over the membrane and $c = T/$ mass/unit area. We assume c^2=constant. The usual boundary condition for a finite membrane) is that $w = 0$ on the boundary.

Simple Harmonic Vibrations

$$w(x, y, t) = W(x, y)\cos(wt + \epsilon) \tag{2}$$

Then

$$\nabla_1^2 W + k^2 W = 0 \qquad\qquad k^2 = \frac{w^2}{c^2} \tag{3}$$

For a circular membrane (complete or annular) we use plane polar coordinates r, θ

$$\nabla_1^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + k^2 W = 0$$

This is separable. i.e. we can find solutions of the form $W(r\theta) = F(r)G(\theta)$ by substitution we have

$$\frac{1}{F}\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + k^2 + \frac{1}{r^2}\frac{1}{G}\frac{d^2G}{d\theta^2} = 0$$

Therefore

$$\frac{1}{G}\frac{d^2G}{d\theta^2} = \text{constant} = -n^2 \tag{4}$$

Therefore

$$G(\theta) = A\cos n\theta + B\sin n\theta$$

$$r\frac{d}{dr}\left(r\frac{dF}{dr}\right) + (k^2r^2 - n^2)F = 0 \tag{5}$$

[Note that $\frac{d}{dr}(r\frac{dF}{dr}) + (k^2r - \frac{n^2}{r})F = 0$ is the self adjoint for ?????????]

In (5) write $kr = x$ (not the co-ordinate)

$$x\frac{d}{dx}\left(x\frac{dF}{dx}\right) + (x^2 - n^2)F = 0 \tag{6}$$

This is **Bessel's Equation of order n**.

The solution of the original equation $\nabla^2W + k^2W = 0$ must be periodic in θ of period 2π, for otherwise W would not be a one valued function of position. Therefore n is an integer.

Series solution for F

We assume $F \sum_{m=0}^{\infty} a_m x^{m+c}$ where c is to be found.

$\left[\left(x\frac{d}{dx}\right)^2 - n^2\right] x^{(m+c)} = [(m+c)^2 - n^2]x^{m+c}$

$\left[\left(x\frac{d}{dx}\right)^2 - n^2\right] F = \sum_{m=0}^{\infty} a_m[(m+c)^2 - n^2]x^{m+c}$

$x^2F = \sum_{m=0}^{\infty} a_m x^{n+c+2} = \sum_{m=2}^{\infty} a_{m-2}x^{m+c}$

Hence we require $\sum_{m=0}^{\infty} a_m[(m+c)^2 - n^2]x^{m+c} + \sum_{m=2}^{\infty} a_{m-2}x^{m+c} = 0$

this is true if

$a_0(c^2 - n^2) = 0$ Indicial equation

$a_1(\overline{c+1}^2 - n^2) = 0$

$a_m(\overline{m+c}^2 - n^2) + a_{m-2} \quad m = 2, 3, \ldots$
Since $a_0 \neq 0$, $c = \pm n$
In the second, putting $c = n$

$$a_1(2n+1) = 0$$

In the third, putting $c = n$, $m = 3, 5, \ldots$

$$a_3 3(2n+3) + a_1 = 0 \cdots$$

We suppose $n \neq -\frac{1}{2}, -\frac{3}{2}, \ldots$
Then $a_1 = a_3 = \ldots = 0$
For $m = 2, 4, 6, \ldots$ in the third relation
$a_2 2(2+2n) + a_0 \Rightarrow a_2 = \dfrac{-a_0}{2^2 \cdot 1(n+1)}$
$a_4 4(4+2n) + a_2 \Rightarrow a_4 = \dfrac{a_0}{2^4 \cdot 1 \cdot 2(n+1)(n+2)}$
$a_{2m} = \dfrac{(-1)^m a_0}{2^{2m} m!(n+1)\ldots(n+m)}$
Hence we have one solution (taking $c = n$)
$a_0 x^n \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+1)\ldots(n+m)} \left(\frac{x}{2}\right)^{2m}$
The Bessel function $J_n(x)$ is defined by taking $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$

$$\begin{aligned} J_n(x) &= \left(\frac{x}{2}\right)^n \frac{1}{\Gamma(n+1)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+1)\ldots(n+m)} \left(\frac{x}{2}\right)^{2m} \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m} \end{aligned}$$

The series converges for all x, by the ratio test. So $J_n(x)$ is an integral function of x. If L_n is the operation in Bessels equation $L_{-n}(y) = L_n(y)$ since n appears as n^2
$L_n[J_n(x)] = 0$ for all n
Therefore $L_{-n}[j_{-n}(x)] = 0$ i.e. $L_n J_{-n}(x)] = 0$
Hence $J_{-n}(x)$ is a solution.

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1} \left(\frac{x}{2}\right)^{-n+2m}$$

When n is a non-negative integer $\Gamma(m+n+1)$ is infinite from

$m = 0, 1, 2, \ldots, n-1$ since $\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$ Therefore

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1}\left(\frac{x}{2}\right)^{-n+2m}$$

Write $m = n + k$

$$J_{-n}(x) = \sum_{k=o}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{x}{2}\right)^{2k+n} = (-i)^n J_n(x)$$

Wronskin of J_n and J_{-n}

$$W = \begin{vmatrix} J_n & J_{-n} \\ J_n' & J_{-n}' \end{vmatrix}$$

we have $\dfrac{d}{dx}\left(x\dfrac{d}{dx}j_{\pm n}\right) + \left(x - \dfrac{n^2}{x}\right)J_{\pm n} = 0$

Therefore $\dfrac{d}{dx}\left\{x[J_n'J_{-n} - J_nJ_{-n}']\right\} = 0$

i.e. $x[J_n'J_{-n} - J_nJ_{-n}'] = C$ (constant).

$C = \lim_{x\to 0} x[J_n'J_{-n} - J_nJ_{-n}']$

$$\begin{aligned}
J_n &= \frac{\left(\frac{x}{2}\right)^n}{r(n+1)}\left[1 - \frac{\left(\frac{x}{2}\right)^2}{1!(n+1)} + \cdots\right] \\
J_{-n} &= \frac{\left(\frac{x}{2}\right)^{-n}}{r(-n+1)}\left[1 - \frac{\left(\frac{x}{2}\right)^2}{1!(-n+1)} + \cdots\right] \\
xJ_n' &= \frac{n\left(\frac{x}{2}\right)^n}{r(n+1)}\left[1 + 0x^2\right] \\
xJ_{-n}' &= \frac{-n\left(\frac{x}{2}\right)^{-n}}{r(-n+1)}\left[1 + 0x^2\right]
\end{aligned}$$

Therefore $J_{-n} \cdot xJ_n' - xJ_{-n}'J_n = \dfrac{2n}{\Gamma(1+n)\Gamma(1-m)}[1 + 0x^2]$

Therefore $C = \dfrac{2n}{\Gamma(1+n)\Gamma(1-m)} = \dfrac{2}{\Gamma(1-n)\Gamma(n)} = \dfrac{2\sin n\pi}{\pi}$

Therefore $J_{-n} \cdot xJ_n' - xJ_{-n}'J_n = \dfrac{2}{x}\dfrac{\sin n\pi}{\pi}$

Thus J_n and J_{-n} are linearly independent when n is not an integer.

and J_n and J_{-n} are linearly dependent when n is an integer.

Definition of the second solution [Weber]

$$Y_n(x) = \frac{\cos n\pi J_n - J_{-n}}{\sin n\pi}$$

when n tends to an integer the numerator tends to 0 nd the denominator tens to 0. Hence when m is an integer we define $Y_m(x) = \lim_{n \to m} Y_n(x)$ Therefore

$$Y_m(x) = \left.\frac{\frac{\partial}{\partial n}[\cos n\pi J_n - J_{-n}]}{\frac{\partial}{\partial n}\sin n\pi}\right\}_{n=m} = \left.\frac{\pi \sin n\pi J_n + \cos n\pi \frac{\partial}{\partial n} - \frac{\partial}{\partial n} J_{-n}}{\pi \cos \pi n}\right\}_{n=m}$$

$$= \frac{1}{\pi}\left[\frac{\partial}{\partial n}(J_n) - (-1)^m \frac{\partial}{\partial n}(J_{-n})\right]_{n=m}$$

In particular $Y_0(x) = \frac{1}{\pi}\left[\left(\frac{\partial}{\partial n}J_n\right)_{n=0} - \left(-\frac{\partial}{\partial n}J_n\right)_{n=0}\right] = \frac{2}{\pi}\left[\frac{\partial}{\partial n}J_n\right]_{n=0}$

The functions of the second kind of order n. They are unbounded at $x = 0$, for fro the relation between J_n and and other solution of Bessel's equations, say $Y_n(x)$, we have

$x[Y_n J_n' - Y_n' J_n] = C$

$Y_n J_N' - Y_n' J_n = \frac{c}{x}$

If $c \neq 0$ then Y_n and Y_n' can not exist at x=0. Hence the general solution of Bessels equations is

either $A_1 J_n(x) + B_1 J_{-n}(x)$ (n not an integer)

or $A_1 J_n(x) + B_1 Y_n(x)$ (all cases)

A solution bounded at $x = 0$ is necessarily $AJ_n(x)$

Returning to the membrane problem

1. Complete membrane ($0 \leq r \leq a$). Since W must be bounded at $r = 0$, we have $F(r) = AJ_n(kr)$

   and $W(r\theta) = AJ_n(kr)\cos(n\theta + \varepsilon)$ (n integer ≥ 0)

   Since $W = 0$ on $r = a$. $AJ_n(ka) = 0$

   $A = 0$ is trivial therefore $J_n(ka) = 0$.

   This is an equation for the eigenvalues k_1^2, $k_2^2, \ldots$. Hence for a given n the values of k are given by $ka = j_{n1}$, $j_{n2}, \ldots$ where j_{n1}, $j_{n2}, \ldots$ are the positive zeros of $J_n(x)$. The allowed frequencies (frequencies of normal modes) are $\frac{w}{a2pi}$, $w = kc$ Therefore

$$w = \frac{c}{a}(j_{n1},\, j_{n2}, \ldots) \quad n = 0, 1, \ldots$$

2. Annular Membrane $b \leq r \leq a$.

   In this case $r = 0$ is not in the "physical space ". Therefore we must take $F(r) = AJ_n(kr) + BY_n(kr)$

   $F(a) = 0 \;\; F(b) = 0$ give:-

$$\begin{aligned} AJ_n(ka) + BY_n(ka) &= 0 \\ AJ_n(kb) + BY_n(kb) &= 0 \end{aligned}$$

   Hence for non-trivial A, B

$$\begin{vmatrix} AJ_n(ka) & BY_n(ka) \\ AJ_n(kb) & BY_n(kb) \end{vmatrix} = 0$$

Sketch of $J_0\, J_1\, J_2$

$J_0 = 1 = \left(\frac{x}{2}\right)^2 \cdot \frac{1}{(1!)^2} + \left(\frac{x}{4}\right)^4 \frac{1}{(2!)^2} + \cdots$

$J_1 = \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^3 \frac{1}{1!2!} + \left(\frac{x}{2}\right)^5 \frac{1}{2!3!}$

Note also $J_0'(x) = -J_1(x)$

PICTURE

The asymptotic formula for $J_n(x)$ is $J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4} + n\frac{\pi}{2}\right)$

Orthogonal and normal Properties of $\mathbf{J_0}$ ($\mathbf{j_m x}$)

where $j1$, $j_2 \ldots$ are the zeros of $j_0(x)$.
We show that

$$\begin{aligned} \int_0^1 xJ_0(j_m x)J_0(j_p x) &= 0 \qquad\qquad p \neq m \\ &= \frac{1}{2}J_1^2(j_m) \quad p = m \end{aligned}$$

the functions are orthogonal to weight function x over $[a,b]$.
$\dfrac{d}{dx}x\dfrac{d}{dx}J_0(\alpha x)+\alpha^2 xJ_0(\alpha x)=0$
$\dfrac{d}{dx}x\dfrac{d}{dx}J_0(\beta x)+\beta^2 xJ_0(\beta x)=0$
$\dfrac{d}{dx}x\left[J_0(\beta x)\dfrac{d}{dx}J_0(\alpha x)-J_0(\alpha x)\dfrac{d}{dx}J_0(\beta x)\right]=(\alpha^2-\beta^2)xJ_0(\alpha x)J_0(\beta x)$
Therefore $x\left[J_0(\beta x)\dfrac{d}{dx}J_0(\alpha x)-J_0(\alpha x)\dfrac{d}{dx}J_0(\beta x)\right]\Big\}_0^1$
$I(\alpha^2-\beta^2)\int_0^1 J_0(\alpha x)J_0(\beta x)\,dx$
i.e.

$$\begin{aligned}(\alpha^2-\beta^2)\int_0^1 xJ_0(\alpha x)-J_0(\beta x)\,dx &= J_0(\alpha)\beta J_0'(\beta)-J_0(\beta)\alpha J_0'(\alpha) \qquad (1)\\ &= J_0(\beta)J_1(\alpha)-J_0(\alpha)\beta J_1(\beta)\end{aligned}$$

If $\alpha=j_m \quad \beta=j_p \quad m\neq p$

$$\int_0^1 xJ_0(j_m x)J_0(j_p x)dx=0$$

$$\begin{aligned}\int_0^1 xJ_0^2(\alpha x) &= \lim_{\beta\to\alpha}\frac{-\alpha J_0(\beta)J_0'(\alpha)+\beta J_0(\alpha)J_0'\beta}{\alpha^2-\beta^2}\\ &= \left[\frac{\frac{\partial}{\partial\beta}\text{Numerator}}{\frac{\partial}{\partial\beta}\text{Denominator}}\right]_{\beta=\alpha}\\ &= \left[\frac{-j_0'(\beta)\cdot\alpha J_0'(\alpha)\frac{\partial}{\partial\beta}(\beta J_0'\beta)}{-2\beta}\right]_{\beta=\alpha}\\ &= -\frac{\alpha J_0'^2(\alpha)-\alpha J_0^2(\alpha)}{-2\alpha}\left[\frac{\partial}{\partial\beta}(\beta J_0'(\beta))+\beta J_0(\beta)=0\right]\end{aligned}$$

Therefore $\displaystyle\int_0^1 xJ_o^2(\alpha x)=\frac{1}{2}J_1^2(\alpha)+J_0^2(\alpha) \qquad [J_0'=-J_1]$ (2)
Therefore then $\alpha=j_m$
$\displaystyle\int_0^1 xJ_0(j_m x)J_0(j_p x)\,dx=\frac{1}{2}J_1^2(j_m)\delta_{mp}$
It also follows that if f_1', f_2', ... are the zeros of $J_0'(x)$ then

$$\int_0^1 xJ_0(j_m' x)J_0(j_p' x)\,dx=\frac{1}{2}J_0^2(j_m)\delta_{mp}$$

Special Cases of $\int_0^1 \mathbf{J_0(\alpha_n x) f(x) dx}$

In (1) take $\alpha = \alpha n$ (a zero) $\beta \neq \alpha_m \;\; m = 1, 2, \ldots$
Then $\int_0^1 x J_0(\alpha_n x) J_0(\beta x)\, dx = \dfrac{\alpha_n}{\alpha_n^2 - \beta^2} J_1(\alpha_n) J_0(\beta)$ (3)

In this put $\beta = 0$ and $J_0(0) = 1$ and so $\int_0^1 x J_0(\alpha_n x)\, dx = \dfrac{J_1(\alpha_n)}{\alpha_n}$ (4)

(4) can also be obtained as follows:
$\dfrac{d}{dx} x \dfrac{d}{dx} J_0(\alpha_n x) = -\alpha_n^2 x J_0(\alpha_n x)$
Therefore

$$\begin{aligned} -\alpha_n^2 \int_0^1 x J_0(\alpha_n x) dx &= \left[x \frac{d}{dx} J_0(\alpha_n x) \right]_0^1 \\ &= a_n J_0'(\alpha_n) \\ &= -\alpha_n J_1(\alpha_n) \end{aligned}$$

Next we consider

$$I_k = \int_0^1 x J_0(\alpha x) x^k \, dx$$

We have

$$\frac{d}{dx} x \frac{d}{dx} J_0(\alpha x) = -\alpha^2 x J_0(\alpha x) \qquad \text{(i)}$$

$$\frac{d}{dx} x \frac{d}{dx} x^k = k^2 x^{k-1} \qquad \text{(ii)}$$

Therefore

$$\frac{d}{dx} x \left\{ x^k \frac{d}{dx} J_0(\alpha x) - J_0(\alpha x) \frac{d}{dx} x^k \right\} = -J_0(\alpha x) \{ x \alpha^2 x^k + k^2 x^{k-1} \}$$

Therefore by integration over [0,1]

$$-\alpha^2 I_k - k^2 I_{k-2} = \alpha J_0'(\alpha) - k J_0(\alpha)$$

Therefore $I_0 \, I_2, \ldots$ can be found in terms of $J_0(\alpha)$ and $J_1(\alpha)$. I_1 and I can be found in terms of the "Sturve function "

$$\rightarrow \int_0^1 J_0(\alpha x) dx q J_0(\alpha) J_1(\alpha)$$

Formal Fourier - Bessel Explanations

Assume that $f(x)$, defined in [0,1], possesses and expansion

$$f(x) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n x)$$

Then

$$\begin{aligned}
\int_0^1 x J_0(\alpha_m x) f(x) dx &= \sum_{n=1}^{\infty} A_m \int_0^1 x J_0(\alpha_n x) J_0(\alpha_m x) dx \\
&= A_m \int_0^1 x J_0^2(\alpha_m x)\, dx \\
&= A_m \frac{J_1^2(\alpha_m)}{2} \\
A_m &= \frac{2}{J_1^2(\alpha_m)} \int_0^1 x J_0(\alpha_m) f(x)\, dx
\end{aligned}$$

Initial and Boundary Problem for the Vibrating Membrane

We have, for the displacement $w(r,t)$ in radially symmetric vibrations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r} \right) = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \qquad 0 \le r \le a$$

$w(0,t)$ exists and $w(a,t) = 0$

Also $w(r,0) = f(r)$ $\quad 0 \le r \le a$ $\qquad \dfrac{\partial w}{\partial t}(r,0) = 0 \quad 0 \le r \le a$

Choose a - unit of length and choose unit on time so that $c = 1$. Also replace r by x, giving

$$\frac{1}{x}\frac{\partial}{\partial x} x \frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial t^2}$$

$w(0,t)$ exists and $w(1,t) = 0$

Also $w(x,0) = f(x)$ $\qquad \dfrac{\partial w}{\partial t}(x,0) = 0$

Assume $w(x,t) = \sum_{n=1}^{\infty} J_0(\alpha_n x)\phi_n(t)$

This satisfies the boundary conditions

$$\frac{1}{2} J_1^2(\alpha_n)\phi_n(t) = \int_0^1 x w(x,t) J_0(\alpha_n x) dx$$

$$
\begin{aligned}
\frac{1}{2}J_1^2(\alpha_n)\ddot{\phi}_n(t) &= \int_0^1 xJ_0(\alpha_n x)\frac{\partial^2 w}{\partial t^2}(xt)\,dx \\
&= \int_0^1 J_0(\alpha_n x)\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right)\,dx \\
&= \left[J_0(\alpha_n x)x\frac{\partial w}{\partial x}\right]_0^1 - \int_0^1 x\frac{\partial w}{\partial x}\frac{d}{dx}J_0(\alpha_n x)dx \\
&= 0 + \left[-wx\frac{d}{dx}J_0(\alpha_n x)\right]_0^1 + \int_0^1 w\frac{d}{dx}x\frac{d}{dx}J_0(\alpha_n x)\,dx \\
&= 0 + \int_0^1 w[-\alpha_n^2 xJ_0(\alpha_n x)]\,dx \\
&= -\alpha^2\cdot\frac{1}{2}J_1^2(\alpha_n)\phi_n(t)
\end{aligned}
$$

Therefore

$$\ddot{\phi}(t) + \alpha_n^2\phi(t) = 0$$

and

$$\phi_n(t) = A_n\cos(\alpha_n(t) + B_n\sin(\alpha_n t))$$

$$
\begin{aligned}
&w(x,0) = f(x) && 0 \le x \le 1 \\
&\frac{\partial}{\partial t}w(x.0) = 0 && 0 \le x \le i
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x) &= \sum_0^\infty J_0(\alpha_n x)\phi_n(0) \\
0 &= \sum_9^\infty J_0(\alpha_n x)\dot{\phi}_n(0) \\
\dot{\phi}_n = 0, \qquad \phi_n(0)\cdot\frac{1}{2}J_1^2(\alpha_n) &= \int_0^1 xJ_0(\alpha_n x)f(x)\,dx \\
\text{i.e. } B_n = 0 \qquad A_n &= \frac{2}{J_1^2(\alpha_n)}\int_0^1 xJ_0(\alpha_n x)f(x)\,dx
\end{aligned}
$$

Hence we have the series solution for $w(x,t)$

$$w(x,t) = \sum_1^\infty J_0(\alpha_n x)\cos\alpha_n t\frac{2}{J_1^2(\alpha_n)}\int_0^1 yJ_0(\alpha_n y)f(y)\,dy$$

Alternative procedure

Solutions of the differential equation bounded at $x = 0$
are $J_0(kx)[A\cos kt + \sin kt]$
This satisfies the boundary conditions $w(1,t) = 0$ if $J_0(k) = 0$ thus
$k = \alpha_1,\, \alpha_2, \ldots$
The solution also satisfies $\dfrac{\partial w}{\partial t}(x0) = 0$ if $B_n = 0$.
Formally the series $\sum_{n=1}^{\infty} A_n J_0(\alpha_n x)\cos\alpha_n t$ satisfies both boundary conditions
and the initial conditions on $\dfrac{\partial w}{\partial t}$. Therefore as $w(x_0) = f(x)$ thus
$f(x) = \sum A_n J_0(\alpha_n x)$ Therefore

$$A_n \cdot \frac{1}{2} J_1^2(\alpha_n) = \int_0^1 x J_0(\alpha_n x) f(x)\, dx.$$

Example

$$f(x) = 1 - \frac{J_0(kx)}{J_0(k)}$$

k real and $J_0(k) \neq 0$

$$\begin{aligned}
\int_0^1 xJ_0(\alpha_n x)J_0(kx)dx &= \frac{J_0(k)\alpha_n J_1(\alpha_n) - J_0(\alpha_n)kJ_1(k)}{\alpha_n^2 - k^2} \\
&= J_0(k)\frac{\alpha_n J_1(\alpha_n)}{\alpha_n^2 - k^2} \\
k = 0 \text{ gives} \quad \int_0^1 xJ_0(\alpha_n x)dx &= \frac{J_1(\alpha_n)}{\alpha_n} \\
\int_0^1 xJ_0(\alpha_n x)f(x)dx &= J_1(\alpha_n)\left\{\frac{1}{\alpha_n} - \frac{\alpha_n}{\alpha_n^2 - k^2}\right\} \\
&= \frac{k^2 J_1(\alpha_n)}{\alpha_n(k^2 - \alpha_n^2)}
\end{aligned}$$

Hence in this case:

$$w(x,t) = \sum_{n=1}^{\infty} J_0(\alpha_n x)\cos\alpha_n t \cdot \frac{2k^2}{\alpha_n(k^2 - \alpha_n^2)}\frac{1}{J_1(\alpha_n)}$$

Since $\alpha_n = O(n)$ for large N, and $J_1(\alpha_n) = O\left(\dfrac{1}{\alpha_n^{\frac{1}{2}}}\right) = O\left(\dfrac{1}{n^{\frac{1}{2}}}\right)$
Then the coefficient of $J_0(\alpha_n x)\cos(\alpha_n t)$ is $O\left(\dfrac{1}{n^{\frac{5}{2}}}\right)$

Solution of a linear differential equation by definite integral (or a contour integral)

Preliminary Remarks

Consider the differential equation

$$x\phi(D)y + \psi(D)y = 0$$

where

$$\begin{aligned}\phi(p) &= a_0p^n + a_1p^{n-1} + \ldots + a_n \\ \psi(p) &= b_0p^m + b_1p^{m-1} + \ldots + b_m\end{aligned}$$

Seek a solution

$$y = \int_a^b e^{px}K(p)dp h.2inot \quad y = \int_C e^{px}K(p)dp$$

where $K(p)$ is to be found and a, b (or C) are also to be found (and are independent of x.)
Then

$$\begin{aligned}\phi(D)y &= \int_a^b \phi(D)e^{px}K(p)dp \\ &= \int_a^b \phi(p)e^{px}K(p)dp \\ \psi(D)y &= \int_a^b \psi(p)e^{px}K(p)dp \\ \text{then} \qquad x\phi(D)y &= \int_a^b xe^{px}\phi(p)K(p)dp \\ &= \int_a^b \frac{d}{dp}(e^{px})\phi(p)K(p)dp \\ &= [e^{px}\phi(p)K(p)]_a^b - \int_a^b e^{px}\frac{d}{dp}(\phi(p)K(p))dp\end{aligned}$$

Hence

$$x\phi(D)y + \psi(D)y = [e^{px}\phi(p)K(p)]_a^b + \int_a^b e^{px}\left\{\psi(p)K(p) - \frac{d}{dp}\phi(p)K(p)dp\right\}dp$$

(i) Choose $K(p)$ so that the integrand is zero. i.e.

$$\frac{d}{dp}\{\phi(p)K(p)\} = \psi(p)K(p) = \frac{\psi(p)}{\phi(p)}\phi(p)K(p)$$

therefore

$$\phi(p)k(p) = C \exp\left\{ \int^p \frac{\psi(q)}{\phi(q)} dq \right\}$$

Note: if all the zeros of ϕ are simple

$\dfrac{\psi}{\phi} = \sum_1^n \dfrac{a_r}{q - q_r} \Rightarrow \int^p \dfrac{\psi}{\phi} = \log(p - p_r)^{a_r}$

Therefore $K_r = \prod_1^n (p - p_r)^{a_{r-1}}$

(ii) when $K(p)$ is known we choose a and b (or C) so that $[e^{px} K(p)\phi(p)]_a^b = 0$

Consider Bessel's equation on order n

$$\left\{ \left(x\frac{d}{dx} \right)^2 + x^2 - n^2 \right\} y = 0$$

$$x^2 \left\{ \frac{d^2y}{dx^2} + y \right\} + x\frac{dy}{dx} - n^2 y = 0$$

Substitute $y = x^n z$

$$\begin{aligned}
x\frac{dy}{dx} &= x^n \left\{ x\frac{d}{dx} + n \right\} \\
\left(x\frac{d}{dx} \right)^2 y &= x^n \left\{ x\frac{d}{dx} + n \right]^2 z \\
&= x^n \left\{ \left(x\frac{d}{dx} \right)^2 + 2nx\frac{d}{dx} + n^2 \right\} z \\
&= x^n \left\{ x^2\frac{d^2}{dx^2} + (2n+1)x\frac{d}{dx} + n^2 \right\} \\
\left\{ \left(x\frac{d}{dx} \right)^2 + x^2 - n^2 \right\} y &= x^n \left\{ x^2\frac{d^2}{dx^2} + (2n+1)x\frac{d}{dx} + x^2 \right\} z \\
&= x^{n+1} \left\{ x\left(\frac{d^2}{dx^2}{}_1 \right) + (2n+1)\frac{d}{dx} \right\} z
\end{aligned}$$

Hence y satisfies Bessel's equation if z satisfies

$$\left\{x\left(\frac{d^2}{dx^2}+1\right)+(2n+1)\frac{d}{dx}\right\}z=0$$

Consider a solution for z of the form

$$\int_a^b e^{itx}K(t)dt$$

Then
$$\left\{x\left(\frac{d^2}{dx^2}+1\right)+(2n+1)\frac{d}{dx}\right\}\int_a^b e^{itx}K(t)dt$$

$$\begin{aligned} &= \int_a^b \{x(1-t^2)+(2n+1)it\}K(t)e^{itx}dt \\ &= \left[\frac{1-t^2}{i}e^{itx}K(t)\right]_a^b + \int_a^b e^{itx}\left[-\frac{d}{dt}\left\{\frac{1-t^2}{i}K(t)\right\}+it(2n+1)K(t)\right]dt \end{aligned}$$

Hence choose $K(t)$ so that

$$\frac{d}{dt}(1-t^2)K(t)=-(2n-1)tK(t)$$

$$K(t)=c(1-t^2)^{n-\frac{1}{2}}$$

hence a solution of Bessel's equation is

$$y=x^n\int_a^b e^{itx}(1-t^2)^{n-\frac{1}{2}}dt$$

if a and b are chosen so that

$$[(1-t^1)^{n+\frac{1}{2}}e^{itx}]_a^b=0$$

Suppose $n>-\frac{1}{2}$ and also x is real and positive
[There is no difficulty if $x=z$ is complex]

Admissible Pairs of limits

(i) $(-1,+1)$
(ii) $(-1,-1+i\infty)$
(iii) $(+1,+1+i\infty)$
PICTURE

These integrals must be linearly dependent since that are solutions of a second order equation. With proper specification of $(1-t^2)^{n-\frac{1}{2}}$ the relation is 1 = (ii) - (iii)
Consider

$$\int e^{itx}(1-t^2)^{n-\frac{1}{2}}dt$$

round the contour shown.
PICTURE
We choose that branch of $(1-t2)^{n-\frac{1}{2}}$ which is real and positive on AA'. i.e. $\arg(1-t)=0$, $\arg(1+t)=0$ on AA'. AS t passes from a to b round $AB\arg(1-t)$ decreases by $\frac{\pi}{2}$; as t passes from A' to B' round $A'B'\arg(1+t)$ increases by$\frac{\pi}{2}$. Since the integrand $e^{itx}(1-t^2)^{n-\frac{1}{2}}$ is now one-valued and regular on and inside the countour, by Cauchy's Theorem we have

$$\int_{\{A'A\}} = \int_{\{A'A\}} + \int_{\{B'C'\}} + \int_{\{CB\}} + \int_{\{C'C\}} + \int_{\{A'B'\}} + \int_{\{BA\}}$$

We show

(a) $\lim_{h\to\infty}\int_{\{C'C\}} = 0$

(b) $\lim_{\epsilon\to\infty}\int_{\{A'B'\}} = \lim_{\epsilon\to\infty}\int_{\{BA\}} = 0$

We shall then have

$$\int_{-1}^{1} e^{itx}(1-t^2)^{n-\frac{1}{2}}dt = \int_{-1}^{-1_i\infty} e^{itx}(1-t^2)^{n-\frac{1}{2}} - \int_{1}^{1+i\infty} e^{itx}(1-t^2)^{n-\frac{1}{2}}$$

i.e.

$$(i) = (ii) - (iii)$$

since the limits if the three integrals exist as $\epsilon\to\infty$ if $n>-\frac{1}{2}$ and as $h\to\infty$.
on CC', $|1-t^2| = PX\cdot PX' \leq C'X\cdot CX' = h^2+4$
$|e^{itx}| = |e^{ix(u+ih)}| = e^{-x+h}$
Therefore $|\int_{CC'}| \leq e^{-xa}(h^2+4)^{n-\frac{1}{2}}\cdot 2 \to 0$ as $h\to\infty$
on AB we have $t-1=\epsilon e^{t\theta}$ Therefore $|t-1|^{n-\frac{1}{2}} = \epsilon^{n-\frac{1}{2}}$
$e^{itx}(t^2-1)^{n-\frac{1}{2}}$ is bounded in the neighbourhood of $t=1$ with bound M say.
Therefore $|e^{itx}(t^2-1(^{n-\frac{1}{2}}| \leq M\epsilon^{n-\frac{1}{2}}$
Thus $\left|\int_{AB}\right| \leq M\epsilon^{n-\frac{1}{2}}\cdot\frac{\pi}{2}\epsilon\frac{M\pi}{2}e^{n-\frac{1}{2}} \to 0$ as $\epsilon\to 0$
Similarly $\int_{B'A'} \to 0$ as $\epsilon\to 0$ $(n>-\frac{1}{2})$

Hence we have the following solutions of Bessel's equation

(i) $x^n \int_0^1 e^{itx}(1-t^2)^{n-\frac{1}{2}}dt$

(ii) $x^n \int_{-1}^{-1+i\infty} e^{itx}(1-t^2)^{n-\frac{1}{2}}dt$

(iii) $x^n \int_{+1}^{+1+i\infty} e^{itx}(1-t^2)^{n-\frac{1}{2}}dt$

Series Expansions of (i)

$$(i) = x^n \int_{-1}^{1}(i-t^2)^{n-\frac{1}{2}}\sum_0^\infty \frac{(itx)^m}{m!}dt$$

the series for all x, t absolutely and uniformly.

$$\begin{aligned}
(i) &= x^n \sum_{m=0}^\infty \frac{(ix)^m}{m!}\int_{-1}^{1} t^m(1-t^2)^{n-\frac{1}{2}}dt \\
\int_{-1}^{1} t^m(1-t^2)^{n-\frac{1}{2}}dt &= 0\text{if m is odd} \\
\int_{-1}^{1} t^{2m}(1-t^2)^{n-\frac{1}{2}}dt &= 2\int_0^1 t^{2m}(1-t^2)^{n-\frac{1}{2}}dt \\
&= 2\int_0^1 u^m(1-u)^{n-\frac{1}{2}}\cdot\frac{1}{2}u^{\frac{1}{2}}du \\
&= \int_0^1 u^{m-\frac{1}{2}}(1-u)^{n-\frac{1}{2}}du \\
&= \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+n+1)} \\
\Rightarrow (i) &= x^n \sum_{m=0}^\infty \frac{(ix)^{2m}}{(2m)!}\frac{\gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+n+1)} \\
\frac{\Gamma(m+\frac{1}{2})}{(2m)!} &= \frac{\Gamma(\frac{1}{2})\frac{1}{2}\frac{3}{2}\cdots m-\frac{1}{2}}{1\cdot 2\cdot 3\cdots(2m-1)2m} \\
&= \frac{\Gamma(\frac{1}{2})}{2^{2m}\cdot m!} \\
\Rightarrow (i) &= \Gamma(n+\frac{1}{2})\Gamma\left(\frac{1}{2}\right)x^n\sum_{m=0}^\infty \frac{(-1)^m\left(\frac{x}{2}\right)^{2m}}{m!\Gamma(m+n+1)}
\end{aligned}$$

$$
\begin{aligned}
&= 2^n\Gamma(n+\frac{1}{2})\Gamma\left(\frac{1}{2}\right) J_n(x) \Rightarrow J_n(x) \\
&= \frac{1}{2^n\Gamma(n+\frac{1}{2})\Gamma\left(\frac{1}{2}\right)} \times (i)
\end{aligned}
$$

Hanbel Functions of order n (Bessel functions of the third kind)

Definition

$$
\begin{aligned}
H_n^{(1)}(x) &= -2\left(\frac{1}{2^n\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\right)\cdot(iii) \\
H_n^{(2)}(x) &= 2\left(\frac{1}{2^n\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\right)\cdot(ii)
\end{aligned}
$$

Then

$$
J_n(x) = \frac{1}{2}(H_n^{(1)}(x) + H_n^{(2)}(x))
$$

we also define

$$
Y_n(x) = \frac{1}{2i}(H_n^{(1)}(x) - H_n^{(2)}(x))
$$

Alternative integral representation of $\mathbf{H_n^{(1)}(x)}$ and $\mathbf{H_n^{(1)}(x)}$

In the integral representation for $H_n^{(1)}(x)$, $\arg(1-t) = -\frac{\pi}{2}$ Therefore we write $(1-t) = \eta e^{-\frac{\pi i}{2}(=-i\eta}$. Where η goes from 0 to ∞ through real values as t goes from 1 to $1+i\infty$. Thus

$$
(1-t)^{n-\frac{1}{2}} e^{-\frac{\pi i}{2}(n-\frac{1}{2})}
$$

Also

$$
\begin{aligned}
(1+t) &= 2-(1-t) = 2-\eta e^{-\frac{\pi i}{2}} = 2(1+\frac{i\eta}{2}) \\
\Rightarrow (1+t)^{n-\frac{1}{2}} &= 2^{n-\frac{1}{2}}(1+\frac{in}{2})^{n-\frac{1}{2}} \\
\Rightarrow (1-t)^{n-\frac{1}{2}} &= 2^{n-\frac{1}{2}}\eta^{n-\frac{1}{2}}e^{-\frac{\pi i}{2}(n-\frac{1}{2})}\left(1+\frac{i\eta}{2}\right)^{n-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
e^{itx} &= e^{ix(1+i\eta)} = e^{ix}e^{-x\eta} \\
dt &= -e^{-\frac{\pi i}{2}d\eta} \\
\Rightarrow e^{itx}(1-t^2)^{n-\frac{1}{2}}dt &= 2^{n-\frac{1}{2}}e^{i(x-\frac{n\pi}{2}+\frac{pi}{4})}e^{-x\eta}\eta^{n-\frac{1}{2}}\left(1+\frac{i\eta}{2}\right)^{n-\frac{1}{2}} d\eta \\
\Rightarrow H_n^{(1)}(x) &= \frac{2^{\frac{1}{2}}x^n e^{i(x-\frac{n\pi}{2}-\frac{\pi}{4})}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-x\eta}\eta^{n-\frac{1}{2}}\left(1+\frac{i\eta}{2}\right)^{n-\frac{1}{2}} d\eta \quad (I)
\end{aligned}
$$

In the integral for $H_n^{(2)}(x)\arg(t+1)=\frac{\pi}{2}$ Therefore $(t+1)=\eta e^{\frac{\pi i}{2}}=i\eta$ where η goes from 0 to ∞ as t goes from -1 to $+1+i\infty$ we get similarly

$$H_n^{(2)}(x)=\frac{2^{\frac{1}{2}}x^n e^{-i(x-\frac{n\pi}{2}-\frac{\pi}{4})}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-x\eta}\eta^{n-\frac{1}{2}}\left(1-\frac{i\eta}{2}\right)^{n-\frac{1}{2}}d\eta \quad (I)$$

[Note: when x is real and positive, $H_n^{(1)}(x)$, $H_n^{(2)}(x)$ are complex conjugates. Therefore $\frac{1}{2}[H_n^{(1)}(x)+H_n^{(2)}(x)]$ is real, so is $\frac{1}{2i}[H_n^{(1)}(x)-H_n^{(2)}(x)]$.]

Finally substitute $\eta x=u$ in both integrals. For real and positive u goes from 0 to ∞ as η goes from 0 to ∞

$$H_n^{(1)}(x)=\left(\frac{2}{x}\right)^{\frac{1}{2}}\frac{1}{\Gamma(\frac{1}{2})}\frac{e^{i(x-\frac{n\pi}{2}-\frac{\pi}{4})}}{\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}du$$

$$H_n^{(2)}(x)=\left(\frac{2}{x}\right)^{\frac{1}{2}}\frac{1}{\Gamma(\frac{1}{2})}\frac{e^{-i(x-\frac{n\pi}{2}-\frac{\pi}{4})}}{\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left(1-\frac{iu}{2x}\right)^{n-\frac{1}{2}}du$$

Asymptotic Expansions of $\int_0^\infty \mathbf{e^{-u}u^{n-\frac{1}{2}}}\left(\mathbf{1\pm\frac{iu}{2x}}\right)^{\mathbf{n-\frac{1}{2}}}\mathbf{du}$

We consider the case $n=0$ i.e.

$$\int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1\pm\frac{iu}{2x}\right)^{-\frac{1}{2}}du$$

We apply Taylor's formula

$$f(t)=\sum_{r=0}^{n-1}\frac{f^{(r)}(0)t^r}{r!}+\frac{1}{(n-1!)}\int_0^t(t-s)^{n-1}f^{(n)}(s)ds$$

to the function $(1-t)^{-\frac{1}{2}}$ this gives

$$(1-t)^{-\frac{1}{2}}=\sum_{r=0}^{n-1}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots r-\frac{1}{2}}{r!}t^r+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots n-\frac{1}{2}}{(n-1)!}\int_0^t(t-s)^{n-1}(1-s)^{-n-\frac{1}{2}}ds$$

the last term is

$$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots n-\frac{1}{2}}{(n-1)!}t^n\int_0^1(1-v)^{n-1}(1-tv)^{-n-\frac{1}{2}}dv$$

writing $t=\dfrac{u}{2ix}$

$$\left(1-\frac{u}{2ix}\right)^{-\frac{1}{2}}=\sum_{r=0}^{n-1}\frac{\frac{1}{2}\frac{3}{2}\cdots n-\frac{1}{2}}{r!}\frac{u^n}{(2ix)^n}+r_n\left(\frac{u}{x}\right)$$

$$r_n\left(\frac{u}{x}\right) = \frac{\frac{1}{2}\frac{3}{2}\cdots n-\frac{1}{2}}{(n-1)!}\frac{u^r}{(2ix)^r}\int_0^1 (1-v)^{n-1}\left(1-\frac{u}{2ix}v\right)^{-n-\frac{1}{2}} dv$$

If x is real and positive

$$\left|1-\frac{u}{2ix}v\right| = \left(1+\frac{u^2v^2}{4x^2}\right)^{\frac{1}{2}} \geq 1$$

Therefore

$$\begin{aligned}\left|r_n\left(\frac{u}{x}\right)\right| &\leq \frac{\frac{1}{2}\frac{3}{2}\cdots n-\frac{1}{2}}{n!}\frac{u^n}{(2x)^n}\int_0^1 (1-v)^{n-1}\cdot 1dv \\ &= \frac{\frac{1}{2}\frac{3}{2}\cdots n-\frac{1}{2}}{n!}\frac{u^n}{(2x)^n}\end{aligned}$$

Hence

$$\int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1-\frac{u}{2ix}\right)^{-\frac{1}{2}} du = \sum_{r=0}^{n-1}\frac{\frac{1}{2}\frac{3}{2}\cdots r-\frac{1}{2}}{r!}\frac{1}{(2ix)^r}\int_0^\infty e^{-u}y^{r-\frac{1}{2}}du + R_n(x)$$

$$R_n(x) = \int_0^\infty e^{-u}u^{-\frac{1}{2}}r_n\left(\frac{u}{x}\right) du$$

$$\int_0^\infty e^{-u}u^{r-\frac{1}{2}}du = \Gamma(r+\frac{1}{2} = \Gamma\left(\frac{1}{2}\right)\frac{1}{2}\cdot\frac{3}{2}\cdots r-\frac{1}{2}$$

Also

$$\begin{aligned}|R_n(x)| &\leq \int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left|r_n\frac{u}{x}\right| du \\ &\leq \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots n-\frac{1}{2}}{n!}\frac{1}{(2x)^n}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}du\end{aligned}$$

Therefore

$$\int_0^\infty e^{-u}u^{-\frac{[}{1}}2\left(1-\frac{u}{2ix}\right)^{-\frac{1}{2}} du = \Gamma\left(\frac{1}{2}\right)\sum_{r=0}^{n-1}\frac{[\frac{1}{2}\frac{3}{2}\cdots r-\frac{1}{2}]^2}{r!}\frac{1}{(2ix)^r} + \bar{R}_n\bar{x}$$

Where $|\bar{R}_n| \leq \frac{[\frac{1}{2}\cdot\frac{3}{2}\cdots n-\frac{1}{2}]^2}{n!}\frac{1}{(2x)^n}\cdot\lim_{x\to\infty} x^{n-1} = 0$
There the series is the asymptotic expansion of the left hand side.
$\left(\text{In fact } R_n = 0\left(\frac{1}{x^n}\right)\right)$

Divergence of the Infinite series

The D'Alernbert ratio is

$$\left|\frac{(n+\frac{1}{2})^2}{n}\cdot\frac{1}{2ix}\right| = \frac{1}{2x}\frac{(n+\frac{1}{2})^2}{n}$$

which tends to infity for and x.

Asymptotic Expansion of $\mathbf{H_0^{(1)}(x)\,H_0^{(2)}(x)}$

$$H_0^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{\pi}{4})}\sum_{r=0}^{\infty}\frac{\left[\frac{1}{2}\frac{3}{2}\cdots r-\frac{1}{2}\right]^2}{r!}\frac{1}{(2ix)^r}$$

$$H_0^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x-\frac{\pi}{4})}\sum_{r=0}^{\infty}\frac{\left[\frac{1}{2}\frac{3}{2}\cdots r-\frac{1}{2}\right]^2}{r!}\frac{1}{(2ix)^r}$$

where the remainder after the term in $\dfrac{1}{x^{n-1}}$ has modulas $\leq$ |term in $\frac{1}{x^n}$|
Write
$A(x) = \dfrac{1}{2}\dfrac{1}{\Gamma(\frac{1}{2})}\left[\int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1-\dfrac{u}{2ix}\right)^{-\frac{1}{2}}du + \int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1+\dfrac{u}{2ix}\right)^{-\frac{1}{2}}du\right]$
$B(x) = \dfrac{1}{2}\dfrac{1}{\Gamma(\frac{1}{2})}\left[\int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1-\dfrac{u}{2ix}\right)^{-\frac{1}{2}}du - \int_0^\infty e^{-u}u^{-\frac{1}{2}}\left(1+\dfrac{u}{2ix}\right)^{-\frac{1}{2}}du\right]$
$H_0^{(1)}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{\pi}{4})}[A(x)+iB(x)]$
$H_0^{(2)}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x-\frac{\pi}{4})}[A(x)-iB(x)]$

$$\begin{aligned} J_0(x) &= \frac{1}{2}\left(H_0^{(1)}(x)+H_0^{(2)}(x)\right) \\ &= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[A(x)\cos\left(x-\frac{\pi}{4}\right) - B(x)\sin\left(x-\frac{\pi}{4}\right)\right] \end{aligned}$$

$$\begin{aligned} Y_0(x) &= \frac{1}{2i}\left(H_0^{(1)}(x)-H_0^{(2)}(x)\right] \\ &= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[A(x)\sin\left(x-\frac{\pi}{4}\right) - B(x)\cos\left(x-\frac{\pi}{4}\right)\right] \end{aligned}$$

The general Bessel function of zero order is
$A_1J_0(x)+B_1Y_0(x) = C(\cos xJ_0()+\sin xY_0(x))$

From the definitions of $A(x)$ and $B(x)$

$$A(x) \sim \sum_{r=0}^{\infty} \frac{[\frac{1}{2}\frac{3}{2}\cdots 2r-\frac{1}{2}]^2}{(2r)!}\frac{(-1)^r}{(2x)^{2r}}$$

$$B(x) \sim \sum_{r=0}^{\infty} \frac{[\frac{1}{2}\frac{3}{2}\cdots 2r+\frac{1}{2}]^2}{(2r+1)!}\frac{(-1)^{r+1}}{(2x)^{2r+1}}$$

Zeros of a Bessel Function of zero order

The zeros are given by $\cot(x-\frac{\pi}{4}-\alpha) = \frac{B(x)}{A(x)}$

$$A(x) = 1 + O\left(\frac{1}{x^2}\right) \qquad B(x) = -\frac{1}{8x} + O\left(\frac{1}{x^3}\right)$$

Therefore

$$\frac{A(x)}{B(x)} = -\frac{1}{8x} + O\left(\frac{1}{x^3}\right)$$

Therefore for large x, the zeros are approximately given by $\cot(x-\frac{\pi}{4}-\alpha) = 0$
i.e. $x - \frac{\pi}{4} - \alpha = \left(k+\frac{1}{2}\right)\pi$ k = large integer.
$x = \alpha + \left(k+\frac{3}{4}\right)\pi$

Asymptotic Expansions of $\mathbf{H_0^{(1)}(x)}$ & $\mathbf{H_0^{(2)}(x)}$

$$H_0^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{n\pi}{2}-\frac{\pi}{4})} \sum_{m=0}^{\infty} \frac{(-1)^m(m,n)}{(2ix)^m}$$

$$H_0^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x-\frac{n\pi}{2}-\frac{\pi}{4})} \sum_{m=0}^{\infty} \frac{(-1)^m(m,n)}{(2ix)^m}$$

where $(0,n) = 1$.

$$(m,n) = \frac{(4n^2-1^2)(4n^2-3^2)\cdots(4n^2-(2m-1)^2)}{2^{2m}m!}$$

There expansion are only useful when $x >> n$

Bessel Functions of order $\mathbf{(k+\frac{1}{2})}$ $\mathbf{k = 0, 1, \cdots}$

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\Gamma(n+m+1)}$$

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x \qquad J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{\frac{1}{2}} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(m+\frac{3}{2})}$$

$$\begin{aligned}
2^{2m} m! \Gamma(m+\frac{3}{2}) &= 2^{2m} m! \Gamma(\frac{3}{2}) \frac{3}{2} \frac{5}{2} \cdots m + \frac{1}{2} \\
&= \Gamma(\frac{3}{2}) 2 \cdot 4 \cdots 2m 3 \cdot 5 \cdots (2m+1) \\
&= (2m+1)! \Gamma(\frac{3}{2}) \\
&= (2m+1)! \frac{1}{2} \Gamma(\frac{1}{2}) \\
\Rightarrow J_{\frac{1}{2}}(x) &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m+1)!} \\
&= \left(\frac{2x}{\pi}\right)^{\frac{1}{2}} \frac{\sin x}{x} \\
&= \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x
\end{aligned}$$

Similarly

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x$$

$$\begin{aligned}
H^{(1)}_{\frac{1}{2}} &= -ie^{ix} \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \qquad H^{(2)}_{-\frac{1}{2}} = ie^{-ix} \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \\
H^{(1)}_{k+\frac{1}{2}}(x) &= -2 \left(\frac{x}{2}\right)^{k+\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2}\Gamma(k+1)} \int_1^{1+i\infty} e^{itx} (1-t^2)^k dt \\
&= -2 \left(\frac{x}{2}\right)^{k+\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})k!} \left(1 + \frac{d^2}{dx^2}\right) \int_1^{1+i\infty} e^{itx} dt \\
&= 2 \left(\frac{x}{2}\right)^{k+\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})k!} \left(1 + \frac{d^2}{dx^2}\right) \frac{e^{itx}}{x} \\
&= \frac{e^x}{x^{\frac{1}{2}}} \{\text{Polynimial in } \frac{1}{x}, \text{degree } k\}
\end{aligned}$$

The functions $H^{(1)}_{k+\frac{1}{2}}(x)$, $H^{(2)}_{k+\frac{1}{2}}(x)$ are called spherical Bessel functions. They arise in solution of the wave equation in spherical Polar coordinates.

Radially Progressive Waves in two dimensions

We had for the membrane

$$\nabla_1^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \qquad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and found solutions (in the case of radial symmetry)

$$w = [AJ_0(kr) + BY_0(kr)]\cos(wt+\epsilon) \quad k = \frac{w}{c}$$

assuming the form

$$w = f(r)e^{iwt}$$

(real parts to be taken eventually) we find similar form

$$w = [A_1 H_0^{(1)}(kr) + A_2 H_0^{(2)}(kr)]e^{iwt}$$

since

$$H_0^{(1)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i(kr-\frac{\pi}{4})} \text{ as } r \to \infty$$

$$H_0^{(2)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i(kr-\frac{\pi}{4})} \text{ as } r \to \infty$$

we get

$$H_0^{(1)}(kr)e^{iwt} \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i(w(t+\frac{r}{c})-\frac{\pi}{4})}$$

$$H_0^{(2)}(kr)e^{iwt} \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i(w(t-\frac{r}{c})-\frac{\pi}{4})}$$

The first repents a wave converging to the origin with velocity c, the second a wane diverging from the origin with velocity c.