Constrained Critical Points

Imagine an idealised mountain. The summit will be a critical point (a maximum) for the height function. Consider a path on the mountain which does not pass through the summit. There will be a highest point on this path, but this will not be a critical point of the height function. It is this latter type of point we are concerned with. Suppose the height function is given by $z = f(x, y)$. The projection of the path onto the $x - y$ plane will be a curve with an equation of the form $g(x, y) = 0$. The path itself will be specified by two equations,

Path $= \{(x, y, z) : z = f(x, y) \text{ and } g(x, y) = 0\}$.

To find the highest and lowest points (locally) on the path we are looking for max/min of $f(x, y)$, subject to the constraint $g(x, y) = 0$. The problem can be tackled in two ways.

Method 1: Substitution
Solve $g(x, y) = 0$ for y, giving $y = y(x)$. Then $\Phi(x) = f(x, y(x))$ is a function of one variable.
Example
Find the minimum distance of the line $lx + my + p = 0$ from the origin.
Now distance $= \sqrt{x^2 + y^2}$. This is smallest when $f(x, y) = x^2 + y^2$ is smallest.
Now if $lx + my - p = 0$ then if $m = 0, \quad x = \frac{p}{l}$ and the minimum distance from the origin is $\left|\frac{p}{l}\right|$.
If $m \neq 0, \quad y = \dfrac{(p - lx)}{m}$, so
$\Phi(x) = f(x, y(x)) = x^2 + \dfrac{(p - lx)^2}{m^2} = \dfrac{[(l^2 + m^2)x^2 - 2plx + p^2]}{m^2}$
This is a quadratic with a minimum where $x = \dfrac{pl}{l^2 + m^2}$.
So $y = \dfrac{pm}{l^2 + m^2} \qquad (x^2 + y^2)_{\min} = \dfrac{p^2}{l^2 + m^2}$
So $d_{\min} = \dfrac{|p|}{\sqrt{l^2 + m^2}}$

Method 2: The Method Lagrange Multipliers

It may not be possible to solve $g(x,y)=0$ for y or x. We rely on the following geometrical ideas.

DIAGRAM

Geometrically it is clear that the tangent to the curve must coincide with that of the contour at the constrained critical point. Where the curve crosses a contour it is usually climbing or descending. (There are exceptions which we shall not deal with). At (x,y) the directional derivative of f is given by $f_x \cos\theta + f_y \sin\theta$.

This must be zero since a tangent to a contour is horizontal. Assume that this is not in the y-direction, so $\cos\theta \neq 0$. Thus

$f_x + f_y \tan\theta = 0$.

Now from $g(x,y)=0$ we obtain $g_x + g_y \frac{dy}{dx} = 0$

Now because the tangents coincide, $\frac{dy}{dx} = \tan\theta$, giving $\frac{fx}{gx} = \frac{fy}{gy} = -\lambda$ say,

i.e. $f_x + \lambda g_x = 0 \qquad f_y + \lambda g_y = 0$

We have two equations in 3 unknowns. But $\tan\theta = \frac{dy}{dx}$ only says that the tangents are parallel, not that they are identical. That fact is given by $g(x,y)=0$, giving a third equation.

Now we may not need to find λ if we are only interested in the location (x,y) and λ is sometimes called an undetermined multiplier. The problem of classifying such points analytically is outside the scope of this course. We shall look at problems where max or min is obvious for geometrical reasons. The above discussion gives the reasoning behind the method. To find critical points of $f(x,y)$ subject to $g(x,y)=0$, we solve the three equations

$$\begin{aligned}
\frac{\partial f}{\partial x} + \lambda \frac{\partial y}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 \\
g(x,y) &= 0
\end{aligned}$$

Applying this to the previous problem, we have

$f(x,y) = x^2 + y^2 \qquad g(x,y) = lx + my - p$

We have to solve

$$2x + \lambda l = 0$$

$$\begin{aligned} 2y + \lambda m &= 0 \\ lx + my - p &= 0 \end{aligned}$$

This gives $x = \dfrac{pl}{l^2 + m^2}$ $\qquad y = \dfrac{pm}{l^2 + m^2}$

We shall now consider an example where both f and g are quadratic.

Example
Find the critical points of the function $f(x, y) = x^2 + xy + y^2$ subject to the constraint $g(x, y) = x^2 - xy + y^2 - 1 = 0$

Let $\phi = f + \lambda g$. We require
$\phi_x = 2x + y + \lambda(2x - y) = 0$
$\phi_y = x + 2y + \lambda(-x + 2y) = 0$

i.e. $\quad x(2 + 2y) + y(1 - \lambda) = 0$
$\quad x(1 - \lambda) + y(2 + 2\lambda) = 0 \qquad (A)$

Now $(x, y) = (0, 0)$ is not a solution to $g = 0$, so we must have

$$\begin{vmatrix} 2 + 2\lambda & 1 - \lambda \\ 1 - \lambda & 2 + 2\lambda \end{vmatrix} = 0$$

i.e. $(2 + 2\lambda)^2 - (1 - \lambda)^2 = 0 \qquad (3 + \lambda)(1 + 3\lambda) = 0$
$\lambda = -3$ or $\lambda = -\frac{1}{3}$

$\lambda = -3$
Substituting in (A) gives $-4x + 4y = 0$ i.e. $x = y$
substituting in $g = 0$ gives $x^2 = 1$. So the critical points are $(1, 1)$ and $(-1, -1)$.
$f - 3g = -2(x - y)^2 + 3$ so when $x = y$ we have a maximum value of 3.

$\lambda = -\frac{1}{3}$
Substituting in (A) gives $x\frac{4}{3} + y\frac{4}{3} = 0$ i.e. $x = -y$
substituting in $g = 0$ gives $3x^2 = 1$. So the critical points are $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.
$f - \frac{1}{3}g = \frac{2}{3}(x + y)^2 + \frac{1}{3}$ so when $x = -y$ we have a minimum value of $\frac{1}{3}$.