**Question**

Find the general solution to the differential equation

$$\dot{\mathbf{x}} = M\mathbf{x} \qquad \text{with} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for each of the following matrices $M$:

1. $$M = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \qquad (*)$$

2. $$M = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}$$

3. $$M = \begin{pmatrix} 3 & -5 \\ 2 & 1 \end{pmatrix} \qquad (*)$$

4. $$M = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \qquad (*)$$

**Answer**

1. $$\dot{\mathbf{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x} \qquad \text{so try} \qquad \mathbf{x} = \mathbf{x}_0 e^{\lambda t}$$

   Hence we need $(M - \lambda I)\mathbf{x}_0 = 0$ which implies find $\lambda$ so that $|M - \lambda I| = 0$
   (or else $\mathbf{x}_0 = \mathbf{0}$).
   $\left|\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$ implies $\left|\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}\right| = 0.$
   $(3-\lambda)(-2-\lambda) + 4 = 0$ so that $\lambda^2 - \lambda - 2 = 0$ hence $\lambda = 2, -1$
   For $\lambda = 2$ we must find $\mathbf{x_0}$.

$(M - \lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = D/2$.

Taking $D = 2$, for example gives, $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$.

For $\lambda = -1$ we must find $\mathbf{x_0}$.

$(M - \lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = 2D$.

Taking $D = 1$, for example gives, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$.

The general solution is therefore given by:

$\mathbf{x}(t) = A\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + s\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$.

Or $\quad x = 2Ae^{2t} + Be^{-t} \quad$ and $\quad y = Ae^{2t} + 2Be^{-t}$.

2.

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}\mathbf{x} \qquad \text{so try} \qquad \mathbf{x} = \mathbf{x}_0 e^{\lambda t}$$

Hence we need $(M - \lambda I)\mathbf{x}_0 = 0$ which implies find $\lambda$ so that $|M - \lambda I| = 0$

$\left|\begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$ implies $\left|\begin{pmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{pmatrix}\right| = 0$.

$\lambda(2 + \lambda) - 8 = 0$ so that $\lambda^2 + 2\lambda - 8 = 0$ hence $\lambda = 2, -4$

For $\lambda = 2$ we must find $\mathbf{x_0}$.

$(M - \lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\begin{pmatrix} -2 & 1 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = 2D$.

Taking $D = 1$, for example gives, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$.

For $\lambda = -4$ we must find $\mathbf{x_0}$.

$(M - \lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = -4D$.

Taking $D = 1$, for example gives, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$.

The general solution is therefore given by:
$\mathbf{x}(t) = A\begin{pmatrix}1\\2\end{pmatrix}e^{2t} + B\begin{pmatrix}1\\-4\end{pmatrix}e^{-4t}$.
Or $\quad x = Ae^{2t} + Be^{-4t} \quad$ and $\quad y = 2Ae^{2t} - 4Be^{-4t}$.

3.

$$\dot{\mathbf{x}} = \begin{pmatrix}3 & -5\\2 & 1\end{pmatrix}\mathbf{x} \qquad \text{so try} \qquad \mathbf{x} = \mathbf{x}_0 e^{\lambda t}$$

Hence we need $(M-\lambda I)\mathbf{x}_0 = 0$ which implies find $\lambda$ so that $|M-\lambda I| = 0$
$\left|\begin{pmatrix}3 & -5\\2 & 1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right| = 0$ implies $\left|\begin{pmatrix}3-\lambda & -5\\2 & 1-\lambda\end{pmatrix}\right| = 0.$
$(3-\lambda)(1-\lambda)+10 = 0$ so that $\lambda^2 - 4\lambda + 13 = 0$ hence $\lambda = 2+3i, 2-3i$
For $\lambda = 2+3i$ we must find $\mathbf{x_0}$.
$(M-\lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix}3 & -5\\2 & 1\end{pmatrix} - (2+3i)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x_0\\y_0\end{pmatrix} = 0$
$\begin{pmatrix}1-3i & -5\\2 & -1-3i\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = (1-3i)D/5$.
Taking $D = 5$, for example gives, $\mathbf{x}_1 = \begin{pmatrix}5\\1-3i\end{pmatrix}e^{(2+3i)t}$.
For $\lambda = 2-3i$ we must find $\mathbf{x_0}$.
$(M-\lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left(\begin{pmatrix}3 & -5\\2 & 1\end{pmatrix} - (2-3i)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x_0\\y_0\end{pmatrix} = 0$
$\begin{pmatrix}1+3i & -5\\2 & -1+3i\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = (1+3i)D/5$.
Taking $D = 5$, for example gives, $\mathbf{x}_2 = \begin{pmatrix}5\\1+3i\end{pmatrix}e^{(2-3i)t}$.
The general solution is therefore given by:
$\mathbf{x}(t) = A\begin{pmatrix}5\\1-3i\end{pmatrix}e^{(2+3i)t} + D\begin{pmatrix}5\\1+3i\end{pmatrix}e^{(2-3i)t}$.
To get this into the form of a real solution take $A = a_1 + a_2 i$ and $B = a_1 - a_2 i$ where both $a_1$ and $a_2$ are real numbers so that B is the complex conjugate of A.
In addition use the fact that $e^{(2-3i)t} = e^{2t}(\cos 3t - i\sin 3t)$
Thus $\mathbf{x}(t) = a_1 e^{2t}\begin{pmatrix}10\cos 3t\\2\cos 3t + 6\sin 3t\end{pmatrix} a_2 e^{2t}\begin{pmatrix}10\sin 3t\\6\cos 3t - 2\sin 3t\end{pmatrix}$
Or $\quad x = (10a_1\cos 3t + 10a_2\sin 3t)e^{2t}$
and $\quad y = ((2a_1 + 6a_2)\cos 3t + (6a_1 - 2a_2)\sin 3t)e^{2t}$

4.

$$\dot{\mathbf{x}} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} \qquad \text{so try} \quad \mathbf{x} = \mathbf{x}_0 e^{\lambda t}$$

Hence we need $(M-\lambda I)\mathbf{x}_0 = 0$ which implies find $\lambda$ so that $|M - \lambda I| = 0$

$\left| \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$

$(3-\lambda_(1-\lambda)+1 = 0$ so that $\lambda^2 - 4\lambda + 4 = 0$ hence $\lambda = 2, 2(repeatedroot)$

For $\lambda = 2$ we must find $\mathbf{x_0}$.

$(M - \lambda I)\mathbf{x}_0 = \mathbf{0}$ implies $\left( \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$ implying $x_0 = D$ and $y_0 = D$.

Taking $D = 1$, for example gives, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$.

Now try $\mathbf{x}_2 = (\mathbf{v}_0 + \mathbf{x}_1 t)e^{2t}$ so that

$2(\mathbf{v}_0 + \mathbf{x}_1 t) + \mathbf{x}_0 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} (\mathbf{v}_0 + \mathbf{x}_1 t)$

$2\mathbf{v}_0 + \mathbf{x}_1 - \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{v}_0 = \left( \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - 2I \right) \mathbf{x}_1 t$

And since $\mathbf{x}_1$ satifies the equation given earlier it follows that:

$\left( \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{v}_0 = \mathbf{x}_1$

Hence $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which gives$\mathbf{v}_0 = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$.

Hence $\mathbf{x}_2 = \left( \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) e^{2t}$

The general solution is therefore given by:

$\mathbf{x}(t) = A \left( \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) e^{2t} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^2 t$.

Or $\quad x = (A(2t + 1) + B)e^{2t} \quad$ and $\quad y = (A(2t - 1) + B)e^{2t}$.