Question

i) State Cauchy's integral formula (expressing $f(a)$ as a certian integral around a closed curve surrounding a) paying particular attention to the hypotheses under which the formula holds. State similar formulae for the nth derivatives $f^{(n)}(a)$.

Using these formulae, evaluate

a) $\int_{|z|=1} \frac{\cos z}{z^3} dz$

b) $\int_{|z|=1} \frac{e^z dz}{4z^3 - 12z^2 + 9z - 2}$

ii) Use (i) to prove that if f is analytic in a region A containing a circle γ with centre a and radius R and if $|f(z)| \leq M$ on γ then $|f^{(n)}(a)| \leq \frac{Mn!}{R^n}$.

Deduce Liouville's theorem that a function that is analytic and bounded throughout $\mathbf{C}$ is a constant function.

If $f(z)$ is analytic throughout $\mathbf{C}$ and satisfies for all z with $|z| > R$ an equality $|f(z)| \leq K|z|^{\frac{1}{2}}$, where $K,\ R$ are positive real constants, prove that $f(z)$ is a constant function.

Answer

i) If $f(z)$ is differentiable inside and on a closed contour C, and if a is inside C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Also

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

a) with $f(z) = \cos z$

$$\int_{|z|=1} \frac{\cos z}{z^3} dz = \frac{2\pi i}{2!} f''(0) = \pi i(-\cos 0) = -\pi i$$

b) The denominator factorises as $(z-2)(2z-1)^2$,

so with $g(z) = \dfrac{e^z}{4(z-2)}$

$$\int_{|z|=1} \frac{e^z dz}{(z-2)(2z-1)^2} = \frac{2\pi i}{1!} g'\left(\frac{1}{2}\right)$$

$$g'(z) = \frac{(z-2)e^z - e^z}{4(z-2)^2} = \frac{(z-3)e^z}{4(z-2)^2}$$

so $g'\left(\frac{1}{2}\right) = \dfrac{-\frac{5}{2}e^{\frac{1}{2}}}{4\left(\frac{3}{2}\right)^2} = -\dfrac{5}{18}e^{\frac{1}{2}}$

so $\displaystyle\int_{|z|=1} \frac{e^z dz}{(z-2)(2z-1)^2} = -\frac{5\pi i}{9}e^{\frac{1}{2}}$

ii) Using Cauchy's integral formula and the estimation lemma gives

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R = \frac{Mn!}{R^n}$$

If $|f(z)| \leq M$ for all z then this inequality holds for all R, so $f^{(n)}(a) = 0$ for all n.

Thus the Taylor series for f consists just of the constant term and so $f(z) = f(0)$ for all z. This is Liouville's theorem.

Let a be an arbitrary point of $\mathbf{C}$. Let C be a circle of radius $r > |a|$, with $r > R$.

On C $|f(z)| \leq Kr^{\frac{1}{2}}$

so $|f'(a)| \leq \dfrac{Kr^{\frac{1}{2}}}{r} = \dfrac{K}{r^{\frac{1}{2}}} \to 0$ as $r \to \infty$

Thus $f'(a) = 0$ for all a, so f is constant.