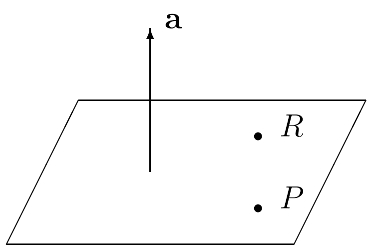
Vector Algebra and Geometry

Geometry of Planes and lines

We assume that each plane has a unique normal direction. If $\mathbf{a} \neq 0$ is a normal to a plane π so is $k\mathbf{a}$ for $k \neq 0$.
Let $\mathbf{a}$ be a non-zero vector $P(\mathbf{p})$ a fixed point and $R\mathbf{r})$ a variable point. Let π be the plane through P perpendicular to $\mathbf{a}$



Then R lies on π if and only if PR is perpendicular to $\mathbf{a}$.
$\vec{PR} = \mathbf{r} - \mathbf{p}$. So R lies on π if and only if $\mathbf{a}(\mathbf{r} - \mathbf{p}) = 0$
i.e. $\mathbf{a} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{p}$
Now if we write $\mathbf{a} \cdot \mathbf{p} = k$ (where k is a constant) then the equation of π can be written as $\mathbf{a} \cdot \mathbf{r} = k$, and if $\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the equation becomes

$$ax + by + cz = k.$$

Notice that the equation of the plane in cartesian form immediately gives us a normal vector (a, b, c).

Example

Find the equation of the plane containing the three points $A(0, 1, -1)$, $B(1, 1, 0)$, $C(1, 2, 0)$
Now $\vec{AB} = (1, 0, 1)$, $\quad \vec{AC} = (1, 1, 1)$
and the normal to the plane is perpendicular to both.
Thus the normal vector is given by $\vec{AB} \times \vec{AC} = (-1, 0, 1)$.
So the equation of the plane is

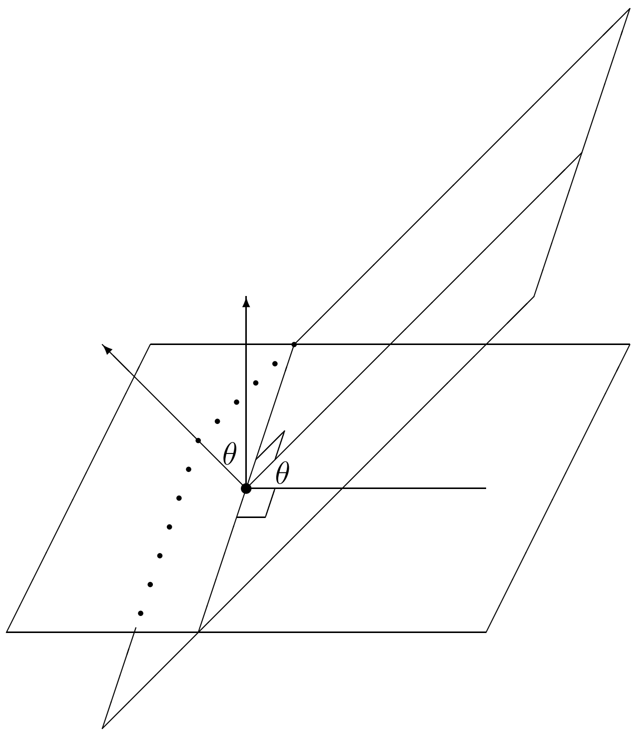
$$-x + z = k.$$

To find K notice that A(0,1,-1) must satisfy the equation of the plane, thus $k = -1$ (check B, C lie in this plane.)

Notice that if AB is parallel to AC then $\vec{AB} \times \vec{AC} = \mathbf{0}$ which is not a normal vector. This corresponds to the situation where ABC are collonear, and there isn't a unique plane containing ABC

Angle between two planes

the angle between the two planes is equal to that between their normal vectors.

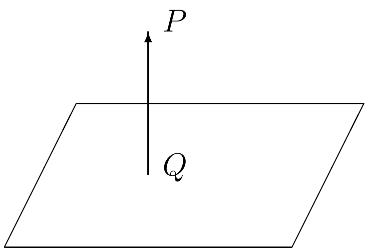


Suppose we have $x + 2y + z = 3$ and $x + y = -4$

then normal vectors are (1,2,1) and (1,1,0) So $\cos\theta = \dfrac{|a \cdot b|}{|a||b|} = \dfrac{3}{\sqrt{6}\sqrt{2}} = \dfrac{\sqrt{3}}{2}$

So $\theta = 30°$

Perpendicular distance
Suppose we have a point $P(\mathbf{p})$ and a plane $\mathbf{a} \cdot \mathbf{r} = k$
Let Q be the foot of the perpendicular, then Q lies in the plane and PQ is parallel to $\mathbf{a}$



So

$$\begin{aligned} \mathbf{a} \cot \mathbf{q} &= k & (1) \\ \mathbf{p} - \mathbf{q} &= t\mathbf{a} & (2) \end{aligned}$$

Now we want $|\mathbf{p} - \mathbf{q}| = |t||\mathbf{a}| = \frac{a \cdot p - k|}{|a|}$

<u>Example</u>
The distance from P(1,2,1) from the plane $x - y + 2z = 5$
($\mathbf{a} = (1, -1, 2)$, $k = 5$)

$$\frac{|(1, -1, 2) \cdot (1, 2, 1) - 5|}{\sqrt{1^1 + 1^2 + 2^2}} = \frac{4}{\sqrt{6}}$$

Distance between parallel planes.
Suppose we have planes $\mathbf{a} \cdot \mathbf{r} = k$ and $\cdot \mathbf{r} = l$. Let P lie in the second plane and Q in the first. The distance between the two planes is

$$PQ = \frac{|\mathbf{a} \cdot \mathbf{p} - k|}{|a|} = \frac{|l - k|}{|a|}$$

as $\mathbf{a} \cdot \mathbf{p} = l$

Equations of a line

We have already met the parametric equation of a line in the form

$$\mathbf{r} = \mathbf{a}t\mathbf{b}$$

representing the line through the point A with position vector $\mathbf{a}$, in the direction of the vector $\mathbf{b}$. Just as the orientation of a plane is specified by a normal vector, so the direction of a line is determined by a direction vector. If $\mathbf{b} \neq 0$ is such a direction vector, so is $k\mathbf{b}$ $(k \neq 0)$. If $\hat{b}$ is a unit direction vector of a line its components are called direction <u>cosines</u> of the line.
Now $\mathbf{r} = (x, y, z)$, $\mathbf{a} = (a, b, c)$, $\mathbf{b} = (l, m, n)$ the equation can be written in terms of components

$$\begin{aligned} x &= a + tl \\ y &= b + tm \\ z &= c + tn \end{aligned}$$

So

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} (= t)$$

These are the cartiesian equations of the line.

<u>Example</u>

Find equation for the line through $P(1, 0, 2)$ and $Q(2, 1, 0)$.
Now a direction vector is $\vec{PQ} = (1, 1, -2)$ $(= \mathbf{b})$
taking $\mathbf{a} = \mathbf{P}$ we obtain

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-2}{-2}$$

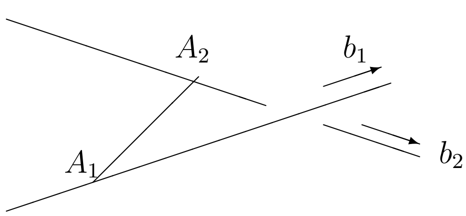
taking $\mathbf{a} = \mathbf{q}$ we obtain

$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z}{2}$$

So we have non-uniqueness in the set of equations.

Two lines in space will, in general, not meet. If they do not meet and are not parallel they are said to be skew.
Suppose we have two lines

$$\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1 \qquad \mathbf{r} = \mathbf{a}_2 + t\mathbf{b}_2$$

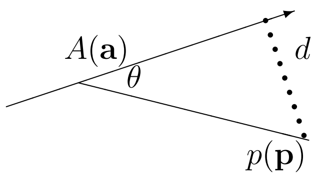


The lines will meet if and only if $\vec{A_1A_2}$, $_1$, b_2 are coplanar. A condition is therefore

$$(a_1 - a_2) \cdot (b_1 \times b_2) = 0$$

Distance of a point from a line

Suppose we hve a line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and a point $\mathbf{p}$ $\mathbf{b}$



$d = AP \sin\theta = |\mathbf{p} - \mathbf{a}| \sin\theta = \frac{|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|}{|\mathbf{b}|}$

Example

Find the distance of the point $P(-1, 2, 1)$ from the line

$$\frac{x-1}{3} = \frac{y}{2} = \frac{z+3}{4}$$

The equation of the line in vector form is

$$\mathbf{r} = (1, 0, -3) + t(3, 2, 4) = \mathbf{a} + t\mathbf{b}$$

So $\mathbf{p} - \mathbf{a} = (-2, 2, 4)$

$(\mathbf{p} - \mathbf{a}) \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & a \\ 3 & 2 & 4 \end{vmatrix} = 0\mathbf{i} + 20\mathbf{j} - 10\mathbf{k}$

So $|(p - a) \times \mathbf{b}| = \sqrt{20^2 + 10^2} = 10\sqrt{5} \quad |\mathbf{b}| = \sqrt{29}$

So the required distance is $\dfrac{10\sqrt{5}}{\sqrt{29}} = 4.152$

Intersections of two planes

Given two planes which are not parallel then we expect them to intersect in a line. We can find the equation of the line in standard from as in the following example.

Let π_1, π_2 be the planes

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x + y + z &= 3 \end{aligned}$$

To find the equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ we need a point on the line $\mathbf{a}$ and a direction vector $\mathbf{b}$. The line will meet at least one of the planes $x = 0$, $y = 0$, $z = 0$ so we try these values in the two equations.

Putting $x = 0$ gives $\begin{matrix} -2 + y = 1 \\ y + z = 3 \end{matrix}$ these give $y = \frac{8}{5}$ and $z = \frac{7}{5}$

So $\mathbf{a} = \left(0, \frac{8}{5}, \frac{7}{5}\right)$ lies in both planes. Now since the line lies in both planes its direction is perpendicular to the normals to both planes and so is parallel to the vector product of these normals.

So we have $\mathbf{b} = (1, -2, 3) \times (2, 1, 1) = (-5, 5, 5)$

So the equation for the line is

$$\mathbf{r} = \left(0, \frac{8}{5}, \frac{7}{5}\right) + t(-5, 5, 5)$$

or in cartisean form

$$-\frac{x}{5} = \frac{y - \frac{8}{5}}{5} = \frac{z - \frac{7}{5}}{5}$$

or

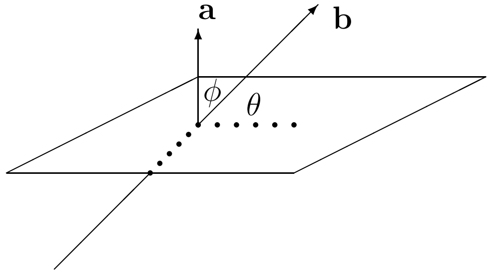
$$-x = y - \frac{8}{5} = z - \frac{7}{5}$$

or

$$-5x = 5y - 8 = 5z - 7$$

Angles between lines

The angle between two lines is defined to be the angle between their direction vectors. The angles between a line and a plane is defined as follows:



If θ is the angle between the line and the plane and $\mathbf{a}$ is the normal to the plane then $\theta = 90 - \phi$

$\cos\phi = \dfrac{a \cdot b}{|a||b|}$

So $\sin\theta \dfrac{a \cdot b}{|a||b|}$

Notice that if the other normal is taken then $a \cdot b < 0$. We want the acute angle so $\sin\theta = \dfrac{a \cdot b}{|a||b|}$

Problems

The following are examples of problems on lines and planes. One should aim to reduce them to some standard results, and so first one needs to think about the sequence of steps to be undertaken.

1. A plane contains the line l_1

$$2x - 2 = 4 - 2y = z - 1$$

and is parallel to the line l_2

$$6x = 3y + 21 = 2z$$

Find its equation.

Firstly to read information about the lines we reduce them to standard form:

$$l_1 : \ \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-2}{2}$$

$$l_2 : \ \frac{x}{1} = \frac{y+7}{2} = \frac{z}{3}$$

If $\mathbf{n}$ is normal to the plane π then $\mathbf{n}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ so we need one point on π, and so any point on l_1 will do.

So $\mathbf{n} = (1, -1, 2) \times (1, 2, 3) = (-7, -1, 3)$

So π has equations $-7x - y + 3z = k$.

Since (1,2,1) lies on l and thus gives $k = -6$ so the required equation is

$$-7x - y + 3z = -6$$

2. Let O be the origin and L be the line

$$x = \frac{y-1}{2} = z - 3$$

Find a point P on l such that OP makes an angle of 45° with L.

The equation of l in vector form is

$$\mathbf{r} = (0, 1, 3) + t(1, 2, 1) = (t, 2t + 1, 3 + t)$$

For the angle between $\vec{OP} = \mathbf{r}$ and the direction $\mathbf{b} = (1, 2, 1)$ to be 45°. We require

$$\mathbf{b} \cdot \mathbf{r} = \frac{|b||r|}{\sqrt{2}}$$

This gives $6t + 5 = \dfrac{\sqrt{6}\sqrt{6t^2 + 10t + 10}}{\sqrt{2}}$

This simplifies to

$$18t^2 + 30t - 5 = 0$$

this gives two values for t, which are then used to give two possible points P

3. A line N contains the point (1,0,2) and meets each of the lines:

$$x = y = z + 2 \qquad - L$$

$$x + 3 = -\frac{y}{2} = \frac{z}{3} \qquad - M$$

Find its equations

Suppose (a,b,c) is a direction vector for N then all popints on N are the form

$$\mathbf{r} = (1 + at,\ bt,\ 2 + ct)$$

For $t = k$ say this point satisfies the equations L

For $t = l$ say the point satisfies the equations M.

This gives four equations in five unknowns a, b, c, k, l.

Eliminating k, l will give

$$\begin{aligned} 4a - 3b - c &= 0 \\ 2a - 5b - 4c &= 0 \end{aligned}$$

Now it is only the ratio $a:b:c$ we need since if (a,b,c) is a direction vector, so is (ma,mb,mc) for any $m \neq 0$. We find that $a:b:c = 1:2:-2$ so that the line N has equations

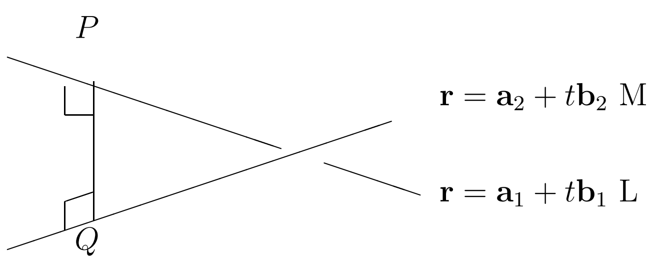
$$\frac{x-1}{1} = \frac{y}{2} = \frac{z-2}{-2}$$

4. Find the shortest distance between the two skew lines

$$L: \quad x-1 = y = z+5$$
$$M: \quad \frac{x}{3} = \frac{y+2}{2} = z+2$$

Find the equations on the common perpendicular and the points where this meets l and m.

Plan 1



Let P be an arbitrary point on L - parameter k

Let P be an arbitrary point on M - parameter l

we want to choose P and Q so that $PQcdotb_1 = 0$ and $PQ \cdot b_2 = 0$ this will give two equations for k and l, which will therefore determine P and Q so we can find $\vec{PQ}$ and from this both $|\vec{PQ}|$ adn the equation of the line (throught P with direct vector $\vec{PQ}$)

solution

$$\begin{aligned}
\mathbf{P} &= (1,0,-5) + k(1,1,1) = (1+k, k, -5+l) \\
\mathbf{Q} &= (0,-2,-2) + l(3,2,1) = (3l, -2-l, 2+l)
\end{aligned}$$

$$\begin{aligned}
\vec{PQ} &= (-1-+3l, -2-k+2lm3-k+l) \\
\vec{PQ}\cdot\mathbf{b}_1 &= -3k+6l = 7 \\
\vec{PQ}\cdot\mathbf{b}_2 &= -4-6+14l = 0
\end{aligned}$$

Solving these equations gives $= 4$ and $l = 2$

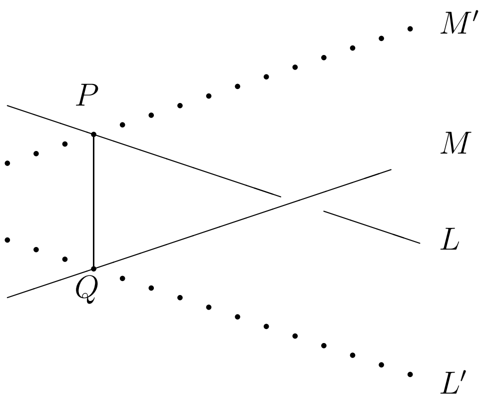
So $P = (5, 4, -1)$ and $Q = (6, 2, 0)$

So $\vec{PQ} = (1, -2, 1) \quad |\vec{PQ} \; = \sqrt{6}$

The equation PQ is

$$\frac{x-5}{1} = \frac{y-4}{-2} = \frac{z+1}{1}$$

<u>Plan 2</u>



Now PL is parallel to QL' and PM' is parallel to QM

Let π_1 be the plane $PQLL'$

Let π_2 be the plane $PQMM'$

1. Work out a direction vector for PQ $\mathbf{c} = \mathbf{b_1} \times \mathbf{b_2}$ will do.
2. Work out the equation of π_1 (contains L and PQ)
3. Work out the equation of π_2 (contains M and PQ)
4. Obtain Q as intersection of π_1 and M
5. Obtain P as intersection of π_2 and L