Vector Algebra and Geometry

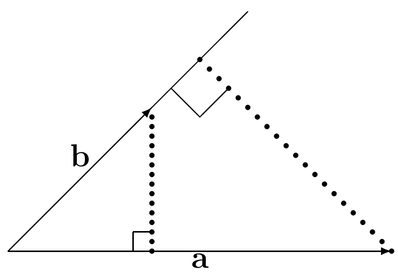
Multiplying Vectors together

When the algebra of vectors was being developed 100 or more years ago many different ways of multiplying vectors were investigated. Two ways have proved most useful. The first, the scalar product, works in all dimensions. The second, the vector product, is an operation particular to three dimensions.

The inner product (often called the scalar product)

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors, and let θ be the angle between them. (Note this can be measured by choosing directed line segments $\vec{PA} = \mathbf{a}$, and $\vec{PB} = \mathbf{b}$, because of the properties of parallels this will be independent of the representatives chosen). We then define the inner product of $\mathbf{a}$ and $\mathbf{b}$ (denoted by $\mathbf{a} \cdot \mathbf{b}$) by

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$



i) Note that $|\mathbf{b}| \cos\theta$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$ and $|\mathbf{a}| \cos\theta$ is the projection of $\mathbf{a}$ onto $\mathbf{b}$.

ii) Note that $\mathbf{a} \cdot \mathbf{b}$ is a number (or scalar).

iii) If $\mathbf{a}$ or $\mathbf{b}$ is a zero we define $\mathbf{a} \cdot \mathbf{b} = 0$

iv) If $\mathbf{a} \cdot \mathbf{b} = 0$ but neither $\mathbf{a}$ nor $\mathbf{b}$ is zero this means that $\cos\theta = 0$

i.e. $\theta = \frac{\pi}{2}$ so $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

v) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos 0 = |\mathbf{a}|^2$

vi) The components (a_1, a_2, a_3) of a vector $\mathbf{a}$ relative to a basis of unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are the projections of $\mathbf{a}$ on $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. So $a_1 = \mathbf{a} \cdot \mathbf{u}_1$ etc.

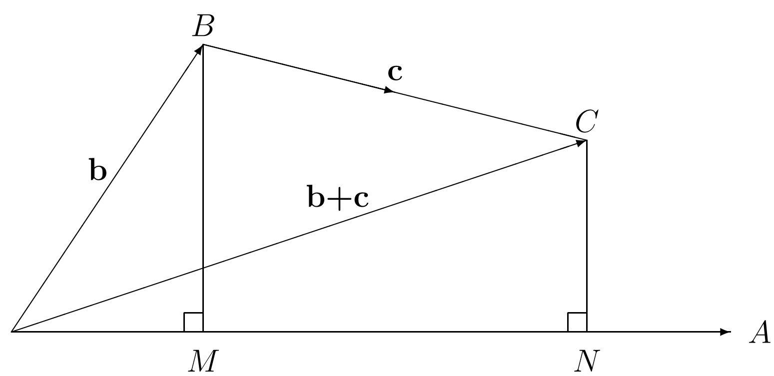
In the alternative notation if $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $x = \mathbf{a} \cdot \mathbf{i}$
$y = \mathbf{a} \cdot \mathbf{j}$, $z = \mathbf{a} \cdot \mathbf{k}$.

We shall first investigate the algebraic properties of this operation.

a) commutative property
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = |\mathbf{b}||\mathbf{a}| \cos\theta = \mathbf{b} \cdot \mathbf{a}$

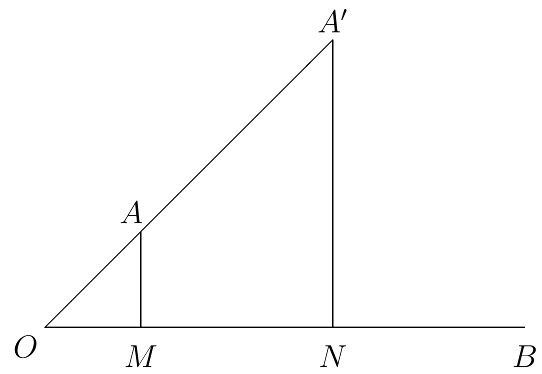
b) distributive properties. We prove these geometrically.

i) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = (\mathbf{b} + \mathbf{c} \cdot \mathbf{a}$ by commutativity).
(Note that the two + signs are different operations).
Choose A, B, C and O so that $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$, $\vec{BC} = \mathbf{c}$.



$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}| \cdot ON = |a| \cdot (OM + MN)$
$= |a| \cdot OM + |a| \cdot MN = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

ii) $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b})$



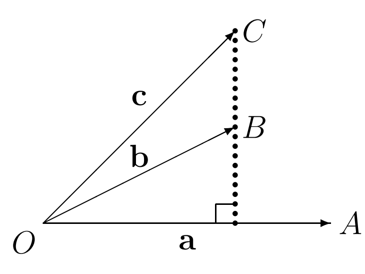
$\vec{OA} = \mathbf{a}$, $\vec{OA'} = k\mathbf{a}$, $\vec{OB} = \mathbf{b}$ by similar triangles $ON = kOM$
so $(k\mathbf{a}) \cdot \mathbf{b} = ON \cdot OB = k \cdot OM \cdot OB = k(\mathbf{a} \cdot \mathbf{b})$

c) In terms of components if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then using the distributive laws.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \mathbf{b} = a_1(\mathbf{i} \cdot \mathbf{b}) + a_2(\mathbf{j} \cdot \mathbf{b}) + a_3(\mathbf{k} \cdot \mathbf{b}) \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

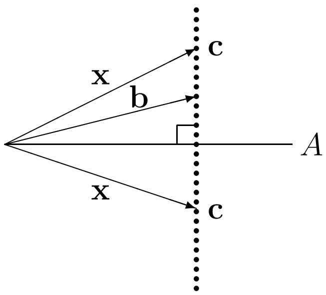
If $\mathbf{b} = \mathbf{a}$ then $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

d) If we have a numerical equation $ax = b$ and $x \neq 0$ then $x = \frac{b}{a}$. However we cannot divide vectors in this way. $ac = bc$, $c \neq 0 \Rightarrow a = b$ for numbers but not for vectors. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ in the diagram, but $\mathbf{b} \neq \mathbf{c}$.



This means that equations do not necessarily have a unique solution for example $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$.

The distributive law gives $\mathbf{a}\cdot(\mathbf{x}-\mathbf{b}) = 0$ then either $\mathbf{x}-\mathbf{b} = 0$ or $\mathbf{x}-\mathbf{b}$ is perpendicular to $\mathbf{a}$. So $\mathbf{x}-\mathbf{b} = \mathbf{c}$ where $\mathbf{c}$ is any vector perpendicular to $\mathbf{a}$ and $\mathbf{x} = \mathbf{b}+\mathbf{c}$.



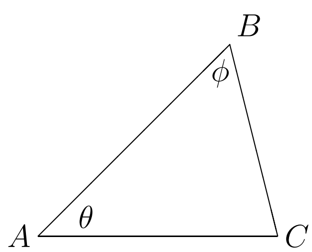
$\mathbf{x}$ can be any vector whose projection on $\mathbf{a}$ is equal to that of $\mathbf{b}$. i.e. we get a plane.

Applications

i) Find the angle between the two lines AB and AC where $A = (1,0,1)$, $B = (2,3,4)$, and $C = (1,3,-2)$.

$\vec{AB} = (2,3,4) - (1,0,1) = (1,3,3)$

$\vec{AC} = (1,3,-2) - (1,0,1) = (0,3,-3)$



$\vec{AB}\cdot\vec{AC} = 1*0 + 3*3 + 3*-3 = 0$

so AB and AC are perpendicular.

To find ϕ, $\vec{BA} = (-1,-3,-3)$, and $\vec{BC} = (1,3,-2) - (2,3,4) = (-1,0,-6)$

$\vec{BA}\cdot\vec{BC} = -1*-1 + -3*0 + -3*-6 = 19 = |BA||BC|\cos\theta$

$|BA|^2 = 1+9+9 = 19$, $|BC|^2 = 1+0+36 = 37$, so $\cos\theta = \dfrac{\vec{BA}\cdot\vec{BC}}{|BA||BC|} = \dfrac{19}{\sqrt{19}\sqrt{37}}$, so $\theta = 44°$.

ii) Find a unit vector which makes an angle of 45° with $\mathbf{a} = (2, 2, -1)$ and an angle of 60° with $\mathbf{b} = (0, 1, -1)$.

Let $\mathbf{u} = (x, y, z)$ be the unknown vector, then we have

a) $|u| = 1$ so $x^2 + y^2 + z^2 = 1$

b) $u \cdot a = |u||a| \cos 45° = |a| \cos 45°$
$2x + 2y - z = \frac{3}{\sqrt{2}}$ $\quad (|a| = 3, \cos 45° = \frac{1}{\sqrt{2}})$.

c) $u \cdot b = |u||b| \cos 60° = |b| \cos 60°$ $\quad y - z = \frac{1}{\sqrt{2}}$

so from b) and c) we obtain

$y = -2x + \sqrt{2} \qquad z = -2x + \dfrac{1}{2}\sqrt{2}$

substituting in a) gives

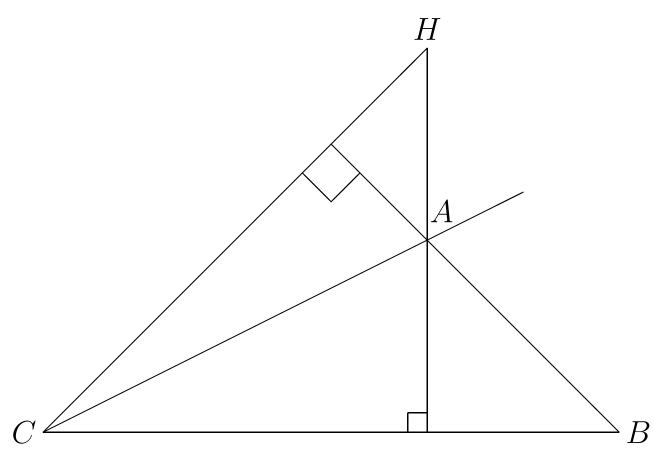
$9x^2 - 6\sqrt{2}x + \frac{3}{2} = 0$

so $x = \frac{1}{\sqrt{2}}$ or $x = \frac{1}{3\sqrt{2}}$ we can now find y and z.

Thus there are two solutions to the problem.

$\mathbf{u} = \left(\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}\right)$ or $\mathbf{u} = \left(\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right)$

iii)



$\vec{HA} = \mathbf{a}$, $\vec{HB} = \mathbf{b}$, $\vec{HC} = \mathbf{c}$. Now HA is perpendicular to BC. So $\mathbf{a}(\mathbf{c} - \mathbf{b}) = 0$ i.e. $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$

Also HC is perpendicular to AB so $\mathbf{c} \cdot (\mathbf{b} - \mathbf{a}) = 0$ i.e. $\mathbf{c} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{a}$

so $\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{b}$ i.e. $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$

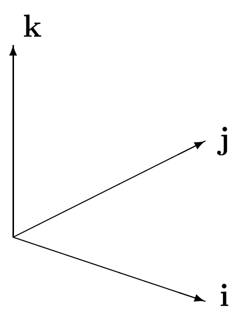
So HA is perpendicular to AC.

This proves that the three altitudes of a triangle are concurrent (at the orthocentre).

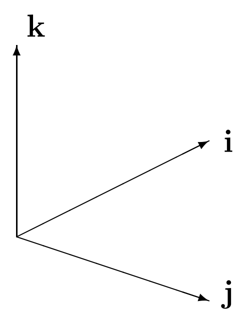
Notice that the choice of origin H simplifies the calculations.

The Vector Product

This product is defined in three dimensions and we need to use the orientation of space



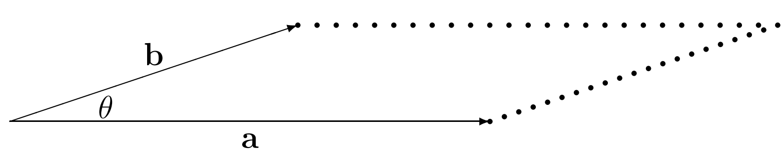
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in that order form a right handed system (rotation $\mathbf{i} \to \mathbf{j}$ clockwise about $\mathbf{k}$).



$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form a left handed system (rotation $\mathbf{i} \to \mathbf{j}$ anti-clockwise about $\mathbf{k}$).

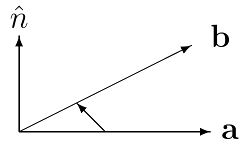
Define the vector $\mathbf{a} \times \mathbf{b}$, with $\mathbf{a}$ and $\mathbf{b}$ not parallel or zero, by
$\mathbf{a} \times \mathbf{b} = |a||b \sin\theta \hat{n}$ where $\hat{n}$ is a unit vector normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\hat{n}$ in that order form a right-handed system.
Define $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if $\mathbf{a}$ and $\mathbf{b}$ are parallel or $\mathbf{a}$ or $\mathbf{b}$ is zero.
Properties

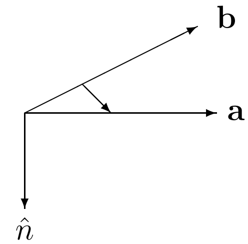
i)



$|a||b|\sin\theta$ is the area of the parallelogram.

ii)



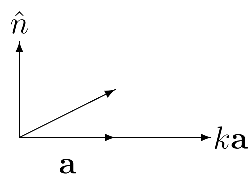


If **a**, **b**, $\hat{n}$ is right-handed, **b**, **a**, $\hat{n}$ is left-handed and **b**, **a**, $-\hat{n}$ is right-handed.

So $\mathbf{b} \times \mathbf{a} = |b||a|\sin\theta(-\hat{n}) = -\mathbf{a} \times \mathbf{b}$

iii) $(k\mathbf{a}) \times \mathbf{b} = |ka||b|\sin\theta\hat{n}$ if $k > 0$

$= k(\mathbf{a} \times \mathbf{b})$



iv) If **a**, **b**, $\hat{n}$ is right-handed, and $-\mathbf{a}$, **b**, $\hat{n}$ is left-handed, so $-\mathbf{a}$, **b**, $-\hat{n}$ is right-handed.

So $(-\mathbf{a}) \times \mathbf{b} = -(\mathbf{a} \times \mathbf{b})$

v) The distributive law

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

$(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = (\mathbf{b} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a})$

The second follows from the first by changing signs.

This turns out to be rather complicated to prove, but as with the scalar product we need it to obtain the separate components of $\mathbf{a} \times \mathbf{b}$.

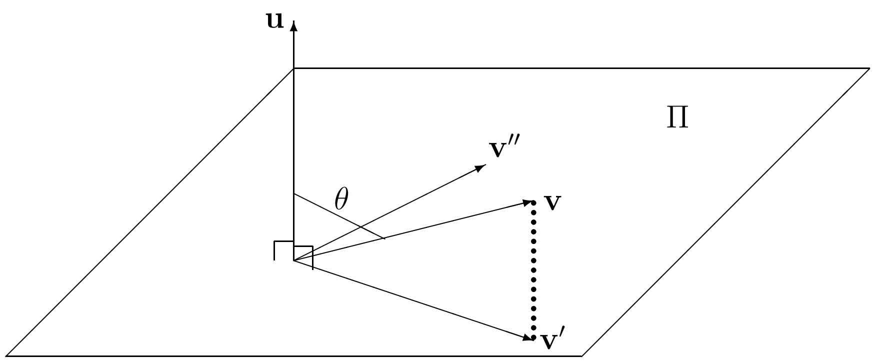
Firstly we write $\mathbf{a} = |\mathbf{a}|\hat{\mathbf{u}}$, so $\hat{\mathbf{u}}$ is a unit vector in the direction of **a**.

by (iii) $\mathbf{a} \times \mathbf{v} = |a|\hat{\mathbf{u}} \times \mathbf{v}$

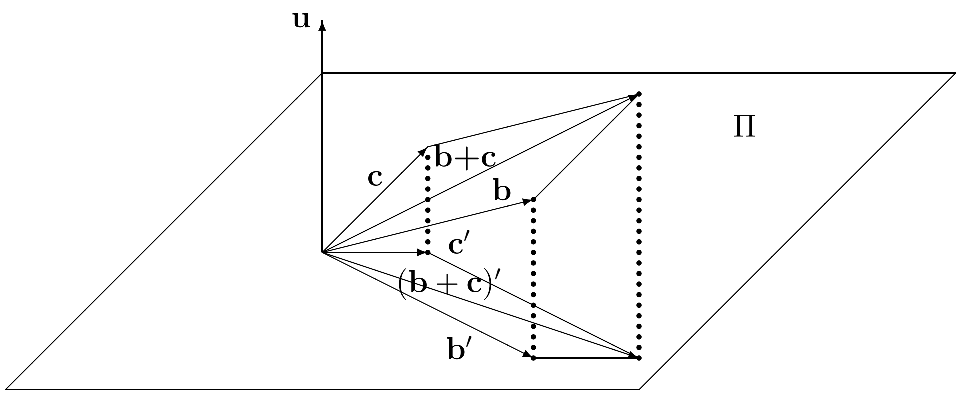
so the distributive rule is proved with $\mathbf{a}$ being a unit vector.

Now consider the plane Π which is normal to $\hat{\mathbf{u}}$. We can obtain $\hat{\mathbf{u}} \times \mathbf{v}$ as follows:

project $\mathbf{v}$ onto Π and rotate the result ($\mathbf{v}'$) through 90° clockwise about $\mathbf{u}$. The result $\mathbf{v}''$ is $\hat{\mathbf{u}} \times \mathbf{v}$. For $|\mathbf{v}''| = |\mathbf{v}'| = |v| \sin\theta$ and $\mathbf{v}''$ is perpendicular to $\hat{\mathbf{u}}$ and $\mathbf{v}$ and $\hat{\mathbf{u}}$, $\mathbf{v}$, $\mathbf{v}''$ forms a right-handed set.



We use this idea to prove the distributive rule.



By the properties of projections, $(\mathbf{b} + \mathbf{c})' = \mathbf{b}' + \mathbf{c}'$. We have

$\hat{\mathbf{u}} \times (\mathbf{b} + \mathbf{c}) = \mathbf{u} \times (\mathbf{b} + \mathbf{c})' = \hat{\mathbf{u}} \times (\mathbf{b}' + \mathbf{c}')$

$(\mathbf{b} + \mathbf{c})'' = \mathbf{b}'' + \mathbf{c}''$

(Rotating the whole parallelogram in the plane Π through 90°).

This proves the result geometrically if $\mathbf{a} \neq \mathbf{0}$. If $\mathbf{a} = \mathbf{0}$ both sides are zero.

This now enables us to find the vector product in component form. We need the products of the various unit vectors **i**, **j**, **k**. We assemble the results in a table.

	i	**j**	**k**
i	**0**	**k**	**-j**
j	**-k**	**0**	**i**
k	**j**	**-i**	**0**

So $\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

This can be written in determinantal form as

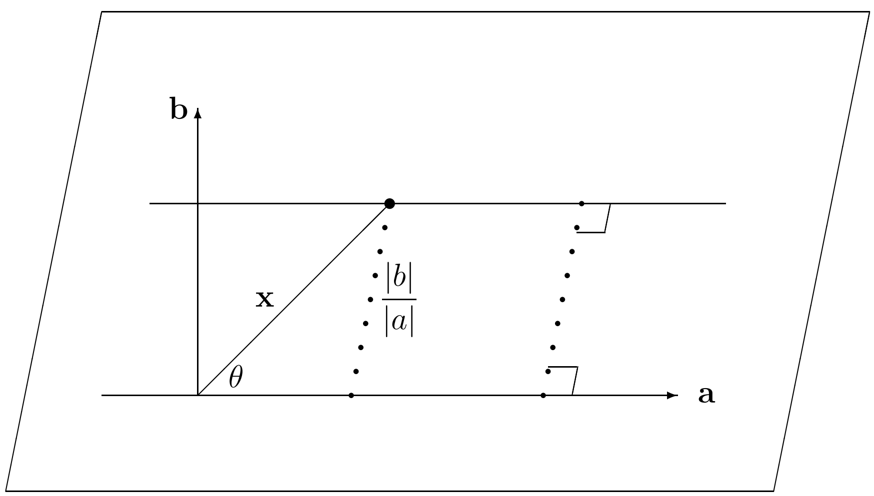
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

vi) Consider now the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ (Interpret as position vectors). This has no solutions unless **a** is perpendicular to **b**. If **a** is perpendicular to **b** then also **x** is perpendicular to **b**, so **x** lies in the plane containing **a** normal to **b**.

$$|a||x| \sin\theta = |b| \qquad \text{so } |x| \sin\theta = \frac{|b|}{|a|} = const.$$

$|x| \sin\theta$ is the distance from **a**.

Again there are many solutions.



As an application of (i) above, consider the triangle ABC with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The area of triangle ABC is half the magnitude of $\vec{AB} \times \vec{AC}$. So it is half of the magnitude of

$$\begin{aligned}(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) &= (\mathbf{b}\times\mathbf{c}) - (\mathbf{b}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{c}) + (\mathbf{a}\times\mathbf{a}) \\ &= (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b})\end{aligned}$$

The area is zero iff ABC are collinear, so a condition for ABC to be collinear is $(\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{b}) = \mathbf{0}$

Products of three vectors

Let us consider the two equations in three unknowns

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \quad (1)$$

$$b_1x_1 + b_2x_2 + b_3x_3 = 0 \quad (2)$$

we can eliminate x_1, x_2, and x_3 in turn to obtain

(3) $\dfrac{x_1}{a_2b_3 - a_3b_2} = \dfrac{x_2}{a_3b_1 - a_1b_3} = \dfrac{x_3}{a_1b_2 - a_2b_1}$ $(= k)$ -some number.

(Assuming none of the denominators are zero.)

These denominators are just the expressions which appear in the vector product. So we consider the above equations vectorially. Suppose we write

$\mathbf{a} = (a_1, a_2, a_3)$ $\mathbf{b} = (b_1, b_2, b_3)$ $\mathbf{x} = (x_1, x_2, x_3)$

(1) and (2) say, $\mathbf{a}\cdot\mathbf{x} = 0$ and $\mathbf{b}\cdot\mathbf{x} = 0$

(3) says $\mathbf{x} = k(\mathbf{a}\times\mathbf{b})$

Thus if $\mathbf{a}\cdot\mathbf{x} = 0$ and $\mathbf{b}\cdot\mathbf{x} = 0$ then $\mathbf{x}$ is parallel to $\mathbf{a}\times\mathbf{b}$. (Assuming $\mathbf{a}\times\mathbf{b} \neq \mathbf{0}$

We now consider

$a_1x_1 + a_2x_2 + a_3x_3 = k$ $\quad\quad$ $\mathbf{a}\cdot\mathbf{x} = k$

$b_1x_1 + b_2x_2 + b_3x_3 = l$ $\quad\quad$ $\mathbf{b}\cdot\mathbf{x} = l$

By eliminating x_1 we obtain

$(a_3b_1 - a_1b_3)x_3 + (a_2b_1 - a_1b_2)x_2 = kb_1 - la_1$

If we write $\mathbf{a}\times\mathbf{b} = \mathbf{d} = (d_1, d_2, d_3)$ then this equation is

$d_2x_3 - d_3x_2 = kb_1 - la_1$

Similarly

$d_3x_1 - d_1x_3 = kb_2 - la_2$

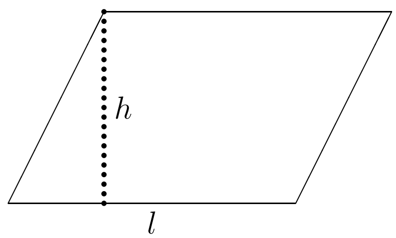
$d_1x_2 - d_2x_1 = kb_3 - la_3$

The left-hand sides are the three components of $\mathbf{d}\times\mathbf{x}$.

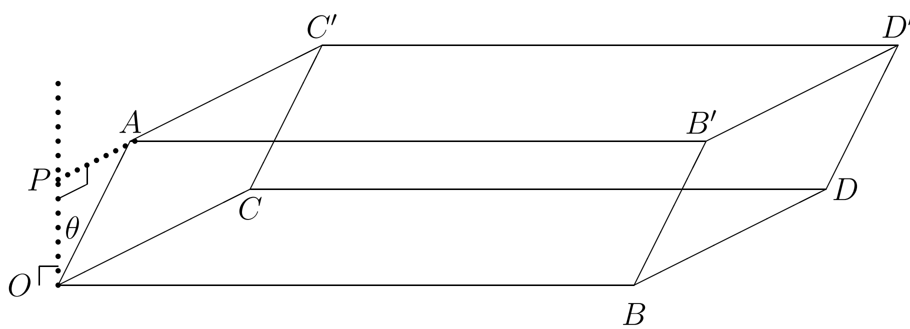
The right-hand sides are the three components of $k\mathbf{b} - l\mathbf{a}$.

So $\mathbf{d}\times\mathbf{x} = k\mathbf{b} - l\mathbf{a}$

i.e. $(\mathbf{a} \times \mathbf{b})) \times \mathbf{x} = (\mathbf{a} \cdot \mathbf{x})\mathbf{b} - (\mathbf{b} \cdot \mathbf{x})\mathbf{a}$
The left-hand side is called a vector triple product. This equation is the vector triple product identity.
Now consider a solid figure determined by pairs of parallel planes in the same way as a parallelogram is determined by pairs of parallel lines. Such a solid is called a parallelepiped.
The area of a parallelogram is the length of a base times the height.



Similarly the volume of a parallelepiped is equal to the area of a base parallelogram multiplied by the corresponding height.



Let OP be the perpendicular distance between the parallel planes $OBDC$ and $AB'D'C'$.
Let O be the origin and ABC have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
Then $\mathbf{b} \times \mathbf{c} =$ area of $OBDC \cdot \hat{OP}$
$|OP| = |a| \cos\theta$
So $V = |a||b \times c| \cos\theta = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
This relies on $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ being a right-handed system. Since the volume is independent of the order in which $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are specified we have

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
$= -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$
In component form we have
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$
$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
This product is called the scalar triple product.
If the volume is zero then this means that $OABC$ are co-planar and the condition is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$
If $OABC$ are co-planar this means that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly dependent.
So $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$ for example, then
$(\alpha\mathbf{b} + \beta\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c})$
$= \alpha(\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c})) + \beta(\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}))$
$= \alpha(\mathbf{c} \cdot (\mathbf{b} \times \mathbf{b})) + \beta(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{c}))$ by cyclic interchange.
$= 0$
If we have four points $PQRS$ then if they are co-planar $\vec{PQ}$, $\vec{PR}$, $\vec{PS}$ are co-planar.
So $\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = 0$