Logic for Web Scientists
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1 Sets

Definition: Set
A set is an unordered collection of objects, without duplicates. A set $A$ containing the objects $a$, $b$ and $c$ is written as $A = \{a, b, c\}$.

Definition: Empty Set
The empty set (written $\emptyset$ or $\{\}$) is the set containing nothing.

Definition: Set Membership
An object $a$ is a member of a set $A$ (written $a \in A$) if it is contained within that collection.

Note: $a \in A$ can be read as “$a$ is a member of $A$” or “$a$ belongs to $A$”.

Definition: Set Equality
Two sets which contain the same objects are considered to be equal (the order of the objects is unimportant).

Example: If $A = \{a, b, c\}$, $B = \{b, a, c\}$ and $C = \{a, b, d\}$, then $A = B$ but $A \neq C$

Definition: Cardinality
The cardinality of a set $A$ (written $|A|$ or $#A$) is the number of members of $A$.

Example: If $A = \{a, b, c\}$, then $|A| = 3$

Definition: Subset
A set $A$ is a subset of a set $B$ (written $A \subseteq B$) if every member of $A$ is also a member of $B$.

Example: If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, c, d\}$, then $A \subseteq B$, but $A \nsubseteq C$
Definition: Strict Subset
A set $A$ is a strict subset of a set $B$ (written $A \subset B$) if every member of $A$ is also a member of $B$, and $A \neq B$.

Example: If $A = \{a, b\}$, $B = \{a, b, c\}$ and $C = \{a, b\}$, then $A \subset B$, but $A \not\subset C$.

Definition: Set Intersection
The intersection of two sets $A$ and $B$ (written as $A \cap B$) is the set containing every object which is both a member of $A$ and a member of $B$.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cap B = \{a, c\}$.

Definition: Set Union
The union of two sets $A$ and $B$ (written as $A \cup B$) is the set containing every object that is a member of $A$ or a member of $B$, or a member of both $A$ and $B$.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A \cup B = \{a, b, c, d\}$.

Mnemonic: $\cup$ stands for U(nion)

Definition: Set Difference
The difference of two sets $A$ and $B$ (written as $A - B$) is the set of every object that is a member of $A$ but not a member of $B$.

Example: If $A = \{a, b, c\}$ and $B = \{a, c, d\}$ then $A - B = \{b\}$

Note: $(A - B) \neq (B - A)$

Definition: Powerset
The powerset of $A$ (written $\mathcal{P}(A)$ or $2^A$) is the set containing all possible subsets of $A$, including $A$ and the empty set.

Example: If $A = \{a, b, c\}$, then $\mathcal{P}(A) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \{\}\}$

Note: $|\mathcal{P}(A)| = 2^{|A|}$.
Definition: Set Comprehension

Rather than explicitly list the members of a set as \( A = \{a_1, \ldots, a_n\} \), we can define a set by specifying the properties that its members must have. This is known as set comprehension.

Set comprehension is expressed using set-builder notation, for which the general form is \( \{x : \phi(x)\} \), where \( x \) is a variable and \( \phi(x) \) is a predicate containing \( x \) which holds true for all members of the set. \( \{x : \phi(x)\} \) can be read as “the set of \( x \) for which \( \phi(x) \) is true”.

Example: \( \{x : x \in \mathbb{Z} \land x > 0\} \)
The set of positive integers - \( \mathbb{Z} \) is the set of integers.
Read as: “the set of \( x \)’s where \( x \) is an integer and \( x \) is greater than zero”.

Example: \( \{x : x \in \mathbb{Z} \land x = x^2\} \)
The set of integers which are equal to their square: \( \{0, 1\} \)

Example: \( \{(x, y) : x \in A \land y \in B\} \)
The set of pairs (\( x, y \)) where \( x \) is a member of set \( A \) and \( y \) is a member of set \( B \). This is the definition of the Cartesian product \( A \times B \) using set-builder notation.

Definition: Tuple

A tuple is an ordered collection of objects, which may include duplicates. The tuple containing \( a, b, c \) and \( a \), in that order, is written \( \langle a, b, c, a \rangle \)

Definition: Arity

The degree or arity of a tuple is the number of objects in the tuple.

Definition: Pair

A tuple containing two objects (a tuple of arity 2) is known as a pair.

Definition: Cartesian Product

The Cartesian product of two sets \( A \) and \( B \) (written \( A \times B \)) is a set of pairs, where each pair contains one member from \( A \) and one member from \( B \), and which contains all possible combinations of members from \( A \) and \( B \).

Example: If \( A = \{a, b, c\} \) and \( B = \{c, d, e\} \), then
\[
A \times B = \{\langle a, c \rangle, \langle a, d \rangle, \langle a, e \rangle, \langle b, c \rangle, \langle b, d \rangle, \langle b, e \rangle, \langle c, c \rangle, \langle c, d \rangle, \langle c, e \rangle\}
\]

Note: \( |A \times B| = |A| \times |B| \)
Definition: Binary Relation

A binary relation \( R \) from set \( A \) to set \( B \) is a set of pairs, where each pair contains one member from \( A \) and one member from \( B \).

Example: If \( A = \{a, b, c\} \) and \( B = \{c, d, e\} \), then a possible relation \( R \) from \( A \) to \( B \) might be:

\[
R = \{ \langle a, c \rangle, \langle a, e \rangle, \langle b, d \rangle, \langle c, c \rangle, \langle c, d \rangle, \} \]

Note: \( R \subseteq A \times B \)

Definition: Domain

The domain of a binary relation \( R \) is the set that the relation goes from.

Example: The domain of \( R \) in the above example is \( A \).

Definition: Range

The range of a binary relation \( R \) is the set that the relation goes to.

Example: The range of \( R \) in the above example is \( B \).

Mnemonic: the range of a cannon is the distance to which it can fire a cannonball.

2 Logic

Definition: Predicate

A predicate is a truth-valued expression. That is, a predicate can either be true or false.

Example: “\( a \in A \)” is a predicate (either \( a \) is a member of \( A \), in which case “\( a \in A \)” is true, or \( a \) is not a member of \( A \), in which case “\( a \in A \)” is false). “\( A \times B \)” is not a predicate, because its value is a set.

Definition: Logical Operators

Predicates may be combined to form compound predicates by using the logical operators: conjunction (\( \land \)), disjunction (\( \lor \)), negation (\( \lnot \)) and implication (\( \Rightarrow \)) .

Definition: Conjunction (logical and)

The conjunction of two predicates \( \phi \) and \( \psi \) (written as \( \phi \land \psi \), and read as “\( \phi \) and \( \psi \)” ) is true if both \( \phi \) and \( \psi \) are true.
Mnemonic: $\land$ stands for A(nd)

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Definition: Disjunction (logical or)

The disjunction of two predicates $\phi$ and $\psi$ (written as $\phi \lor \psi$, and read as “$\phi$ or $\psi$”) is true if either $\phi$ is true or $\psi$ is true (or if both $\phi$ and $\psi$ are true – $\lor$ is the inclusive-or).

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Definition: Negation

The negation of a predicate $\phi$ (written as $\neg \phi$, and read as “not $\phi$”) is true if $\phi$ is false.

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Definition: Implication

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